# A Course Module

## on

# Electromagnetic Field Theory
# (20A02403T)


Prepared by

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# Review of Vector Algebra & Vector Calculus

# ELECTROMAGNETIC FIELD THEORY
## (20A02403T)

# CO-ORDINATE SYSTEMS



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- In order to describe the spatial variations of the quantities, we require appropriate co-ordinate system. A point or vector can be represented in a *curvilinear* coordinate system that may be *orthogonal* or *non-orthogonal.*

- *An orthogonal system is one in which the co-ordinates are mutually perpendicular.*

- Non-orthogonal co-ordinate systems are also possible, but their usage is very limited in practice.

Let

$u$ = constant

$v$ = constant

$w$ = constant

represent surfaces in a coordinate system, the surfaces may be curved surfaces in general.

Let

$$\hat{a}_u \quad \hat{a}_v \quad \hat{a}_w$$

be the unit vectors in the three coordinate directions (base vectors).

In a general right handed orthogonal curvilinear systems, the vectors satisfy the following relations :

These equations are not independent and specification of one will automatically imply the other two.



$$\hat{a}_u \times \hat{a}_v = \hat{a}_w$$
$$\hat{a}_v \times \hat{a}_w = \hat{a}_u$$
$$\hat{a}_w \times \hat{a}_u = \hat{a}_v$$

The following relations hold

$$\hat{a}_u \cdot \hat{a}_u = \hat{a}_v \cdot \hat{a}_v = \hat{a}_w \cdot \hat{a}_w = 1$$

$$\hat{a}_u \cdot \hat{a}_v = \hat{a}_v \cdot \hat{a}_w = \hat{a}_w \cdot \hat{a}_u = 0$$

A vector can be represented as sum of its orthogonal components



$$\vec{A} = A_u \hat{a}_u + A_v \hat{a}_v + A_w \hat{a}_w$$

# Differential Length

- In general $u$, $v$ and $w$ may not represent length. We multiply $u$, $v$ and $w$ by conversion factors $h_1$,$h_2$ and $h_3$ respectively to convert differential changes d$u$, d$v$ and d$w$ to corresponding changes in length d$l_1$, d$l_2$, and d$l_3$. Therefore



$$\overrightarrow{dl} = dl_1\hat{a}_u + dl_2\hat{a}_v + dl_3\hat{a}_w$$

$$\overrightarrow{dl} = h_1du\hat{a}_u + h_2dv\hat{a}_v + h_3dw\hat{a}_w$$

# Differential Area

- Differential area normal to $\hat{a}_u$

$$\overrightarrow{ds_u} = (h_2 dv \hat{a}_v \times h_3 dw \hat{a}_w)$$

$$\overrightarrow{ds_u} = h_2 h_3 dv dw \hat{a}_u$$

- Differential area normal to $\hat{a}_v$

$$\overrightarrow{ds_v} = (h_3 dw \hat{a}_w \times h_1 du \hat{a}_u)$$

$$\overrightarrow{ds_v} = h_1 h_3 du dw \hat{a}_v$$

- Differential area normal to $\hat{a}_w$

$$\overrightarrow{ds_w} = (h_1 du \hat{a}_u \times h_2 dv \hat{a}_v)$$

$$\overrightarrow{ds_w} = h_1 h_2 du dv \hat{a}_w$$

# Differential Volume

$$dv = (h_1 du)(h_2 dv)(h_3 dw)$$

$$dv = h_1 h_2 h_3 du dv dw$$

# Most commonly used Orthogonal Co-Ordinate Systems

1. Cartesian (or rectangular) co-ordinate system

2. Cylindrical co-ordinate system

3. Spherical coordinate system

# CARTESIAN

# (OR RECTANGULAR)

# CO-ORDINATE SYSTEM

In Cartesian Co - Ordinate system, we have,

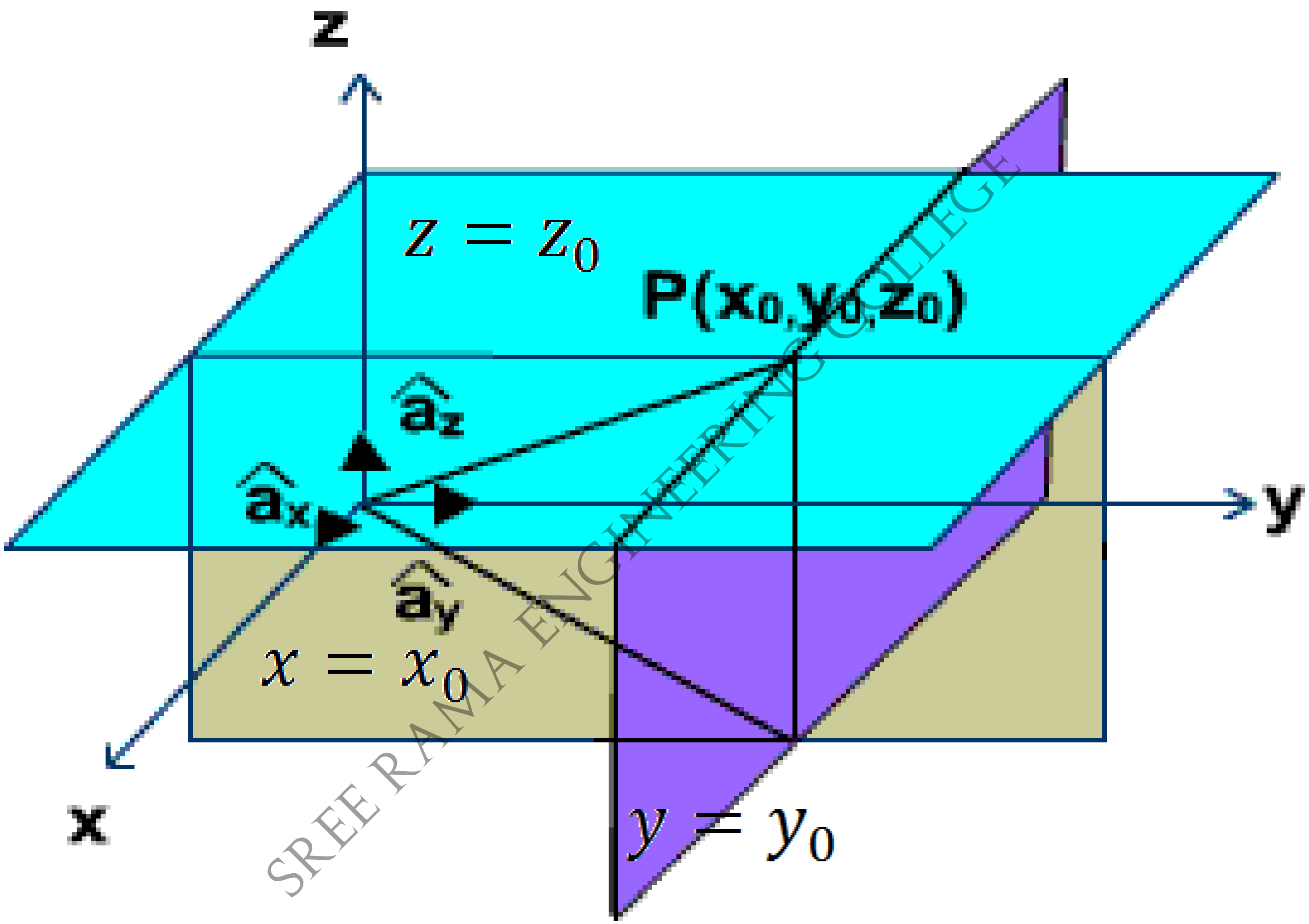
$(u,v,w) = (x,y,z)$.

A point $P(x_o, y_o, z_o)$ in Cartesian co-ordinate system is represented as intersection of three planes

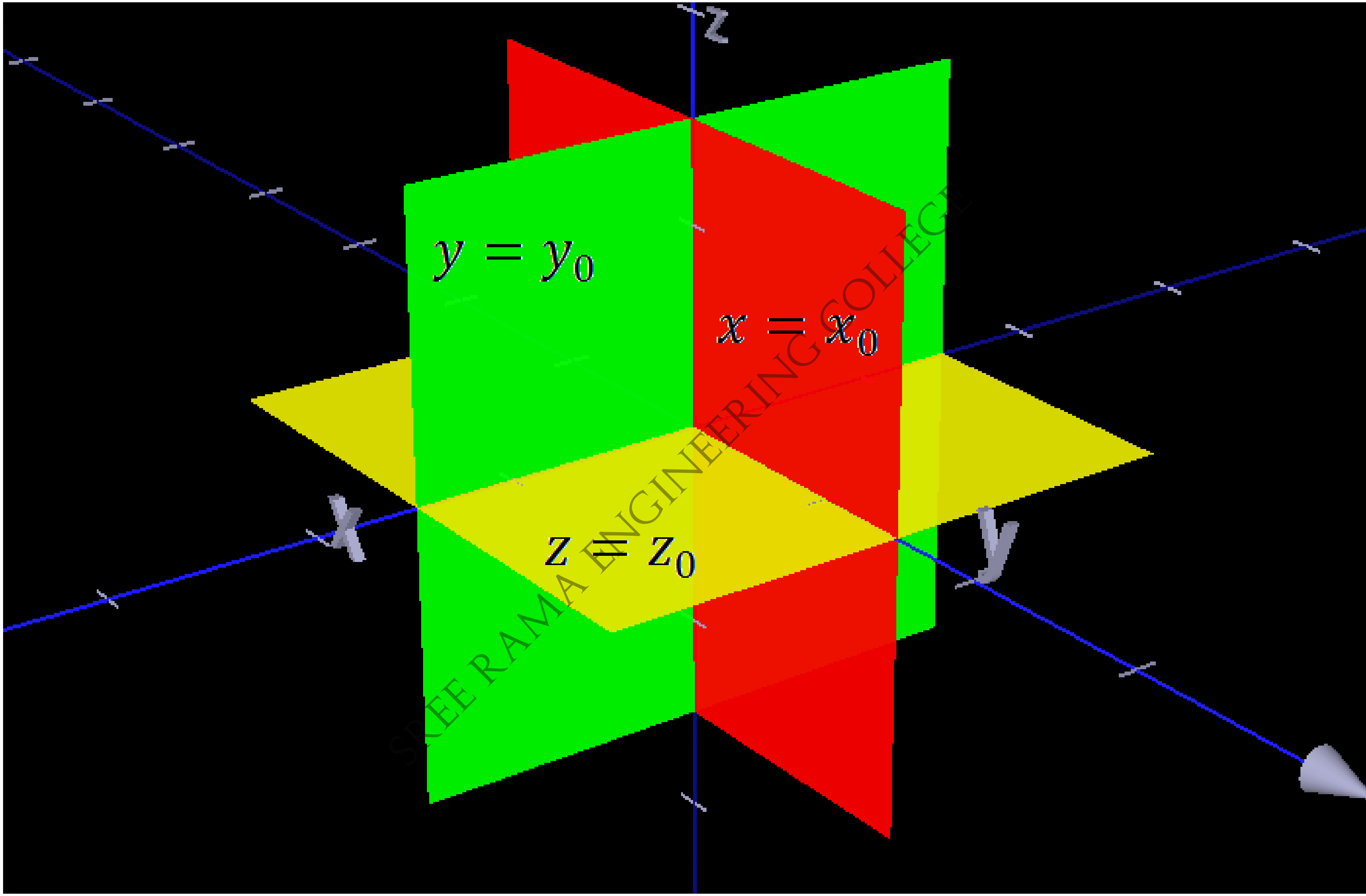
$$x = x_0 \qquad -\infty < x < \infty$$

$$y = y_0 \qquad -\infty < y < \infty$$

$$z = z_0 \qquad -\infty < z < \infty$$



Cartesian Coordinate System


z
$y = y_0$
$x = x_0$
$z = z_0$
x
y
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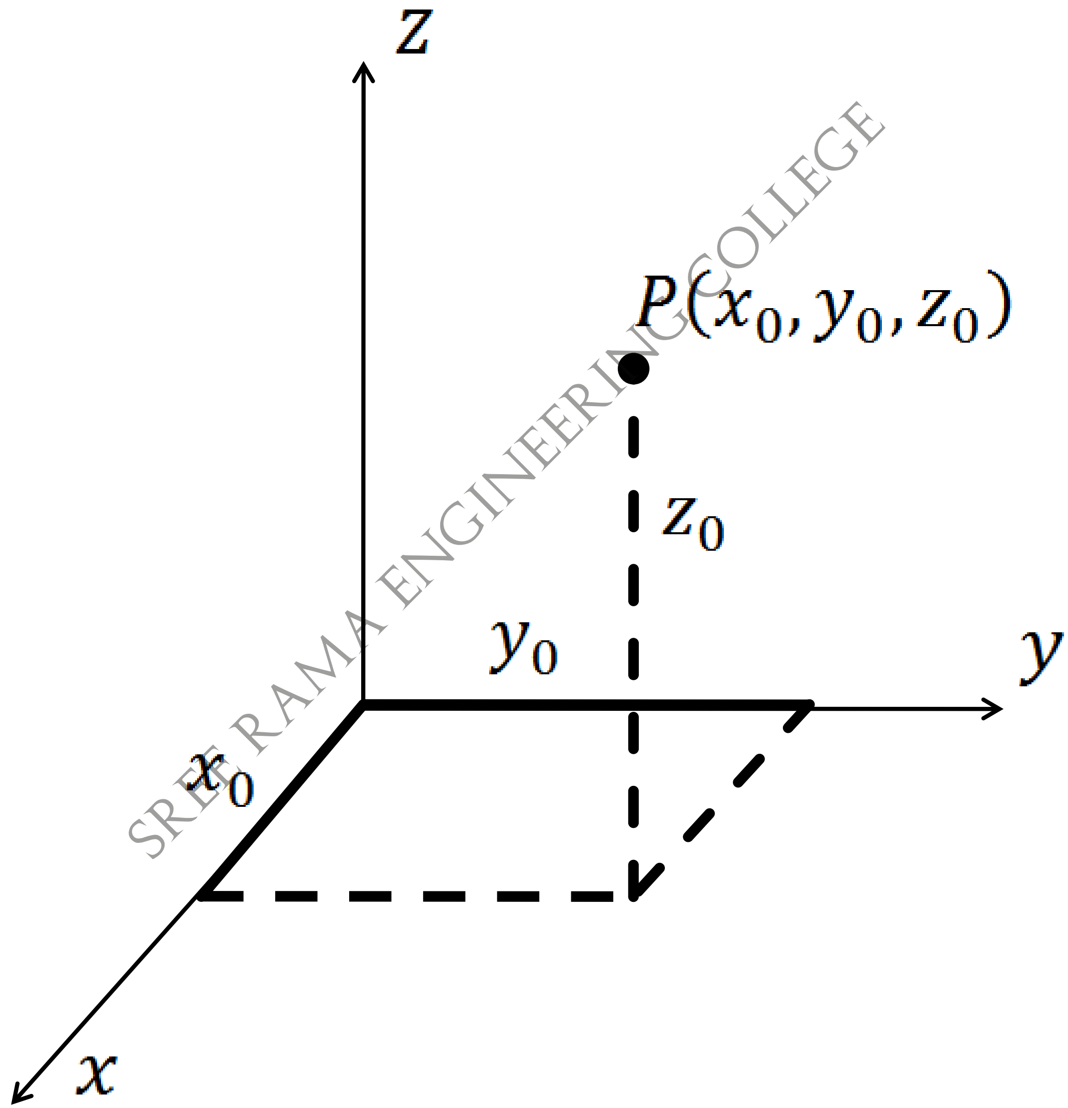
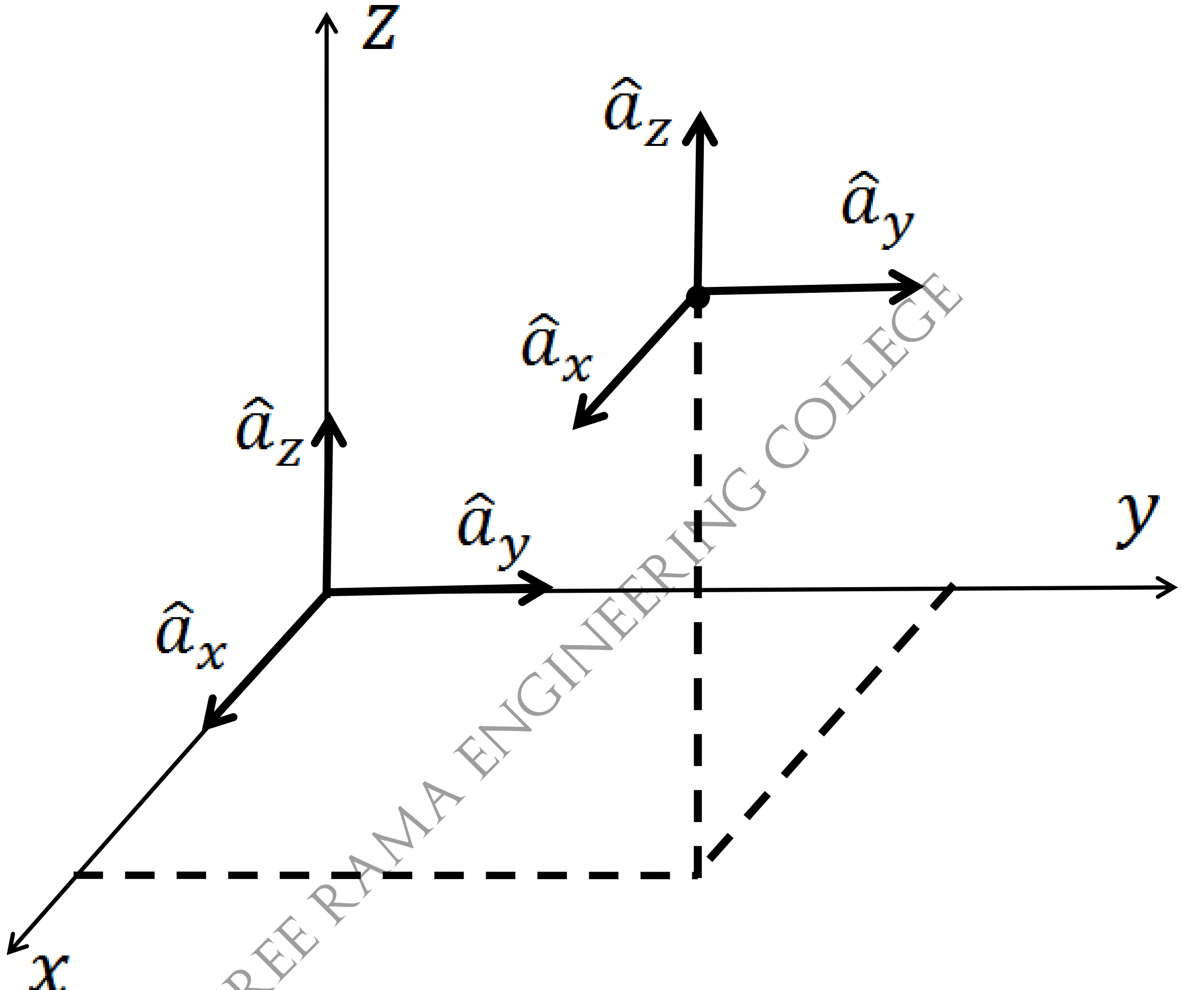
# A point in cartesian coordinate system







$\hat{a}_x, \hat{a}_y \ \hat{a}_z$ are **uniform** unit vectors, that is, the direction of each unit vector is same everywhere in space.

The unit vectors satisfies the following relations:

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z$$

$$\hat{a}_y \times \hat{a}_z = \hat{a}_x$$

$$\hat{a}_z \times \hat{a}_x = \hat{a}_y$$

$$\hat{a}_x \cdot \hat{a}_y = \hat{a}_y \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_x = 0$$

$$\hat{a}_x \cdot \hat{a}_x = \hat{a}_y \cdot \hat{a}_y = \hat{a}_z \cdot \hat{a}_z = 1$$

- In Cartesian co-ordinate system, a vector can be written as

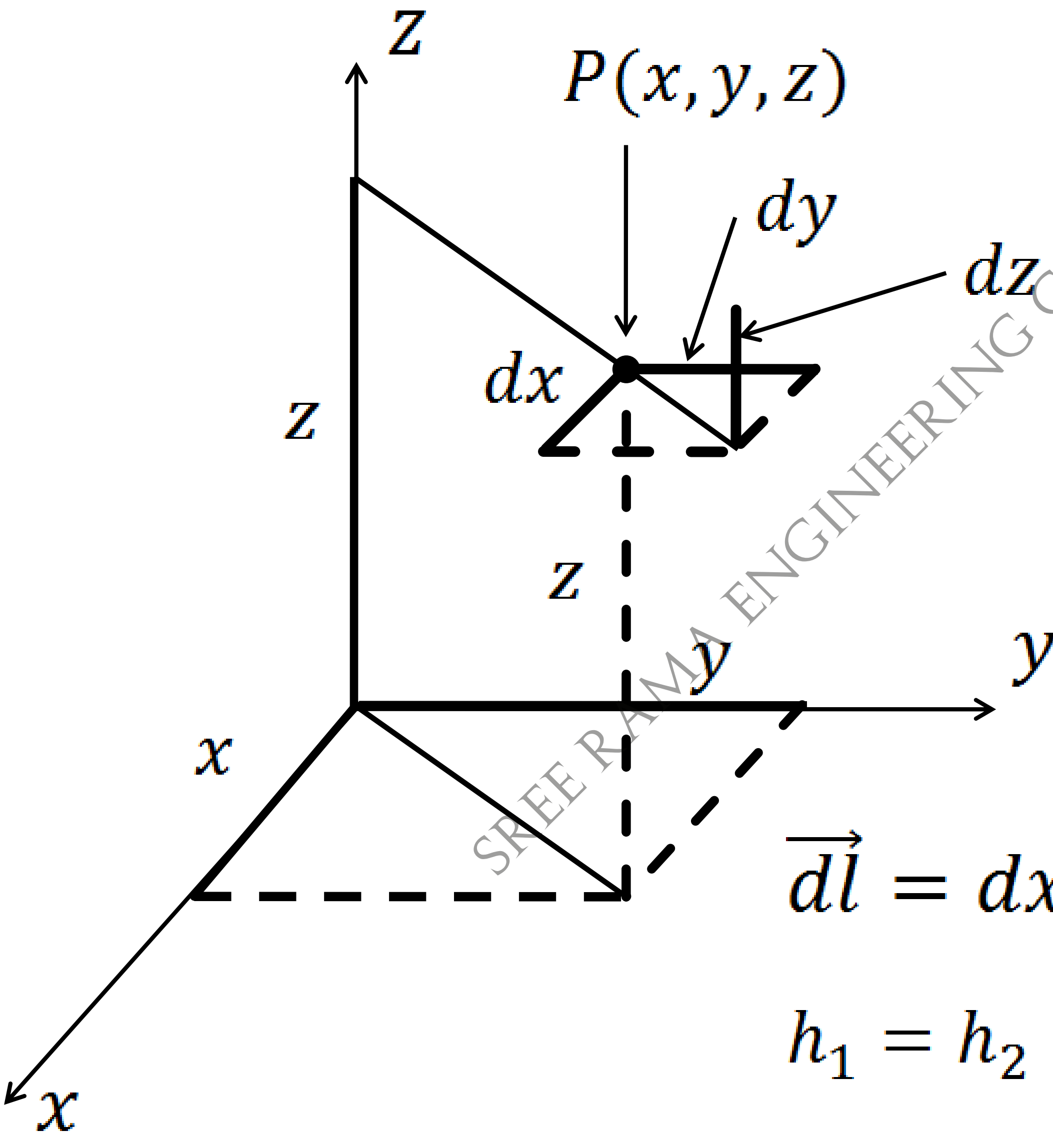
$$\vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$$

- The dot and cross product of two vectors can be written as follows:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

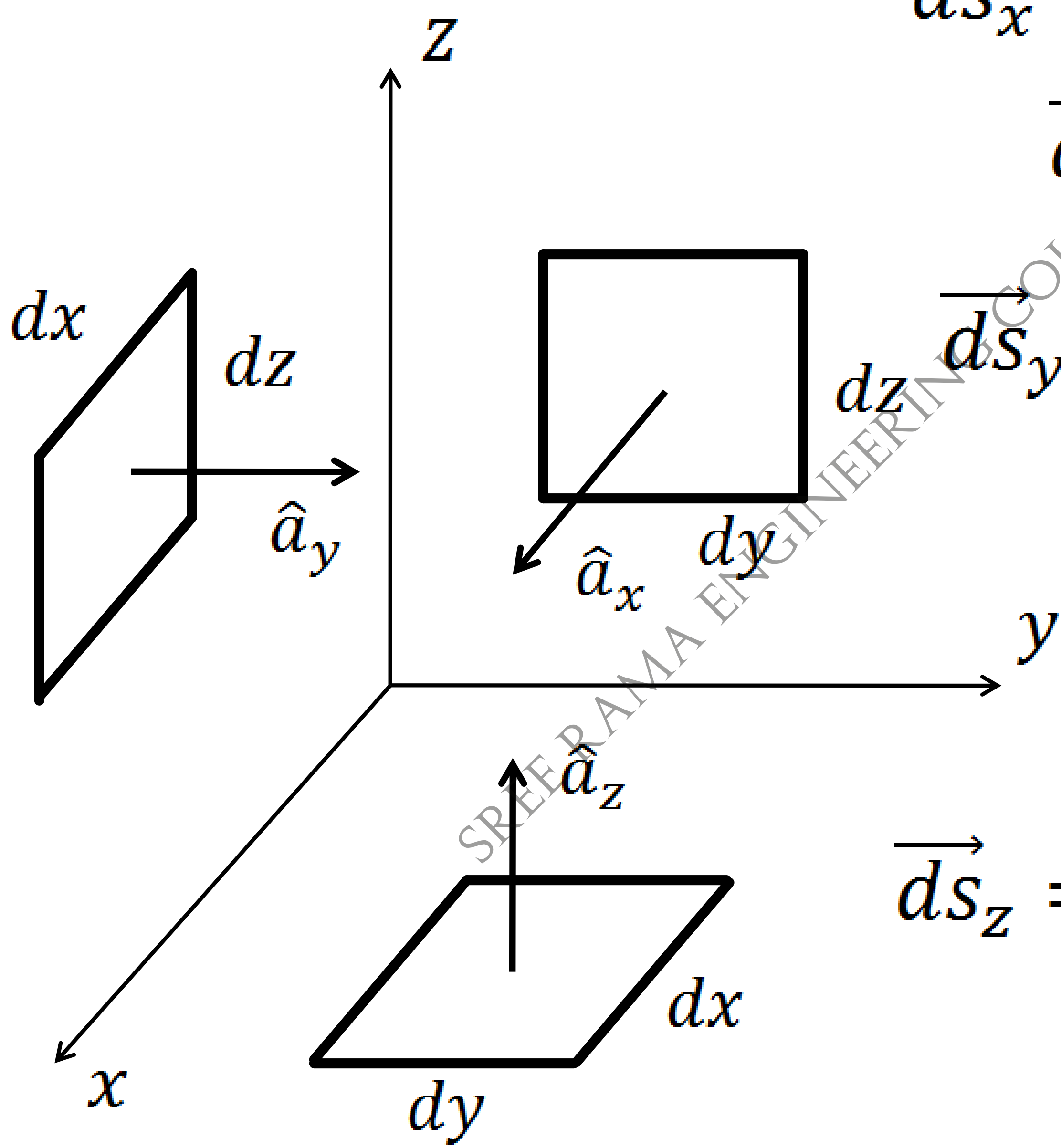
# Differential Length




$$\overrightarrow{dl} = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z$$

$$h_1 = h_2 = h_3 = 1$$

# Differential Area




$$\overrightarrow{ds_x} = (dy\hat{a}_y \times dz\hat{a}_z)$$

$$\overrightarrow{ds_x} = dydz\hat{a}_x$$

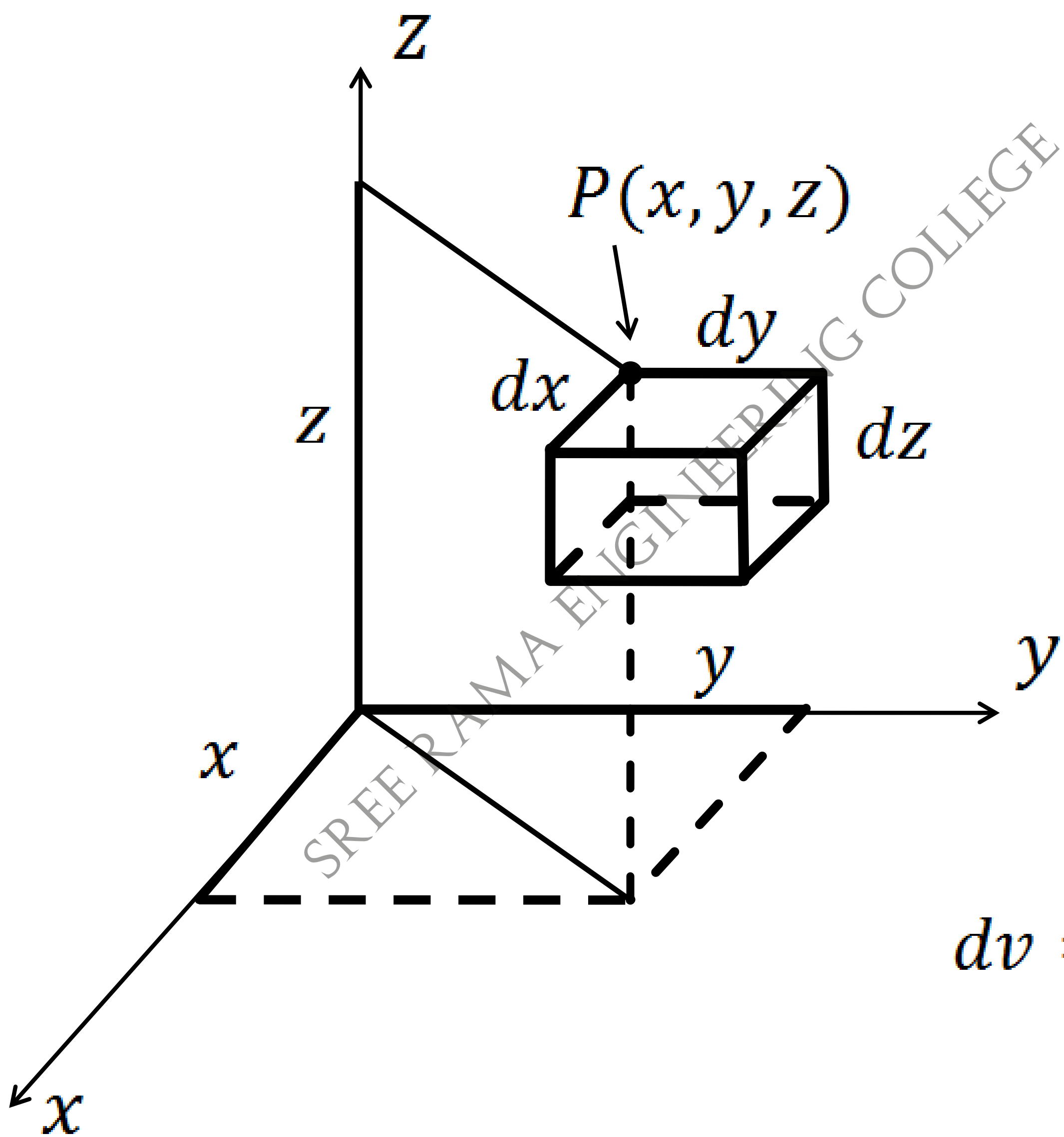
$$\overrightarrow{ds_y} = (dz\hat{a}_z \times dx\hat{a}_x)$$

$$\overrightarrow{ds_y} = dzdx\hat{a}_y$$

$$\overrightarrow{ds_z} = (dx\hat{a}_x \times dy\hat{a}_y)$$

$$\overrightarrow{ds_z} = dxdy\hat{a}_z$$

# Differential Volume




$$dv = dxdydz$$

# CYLINDRICAL

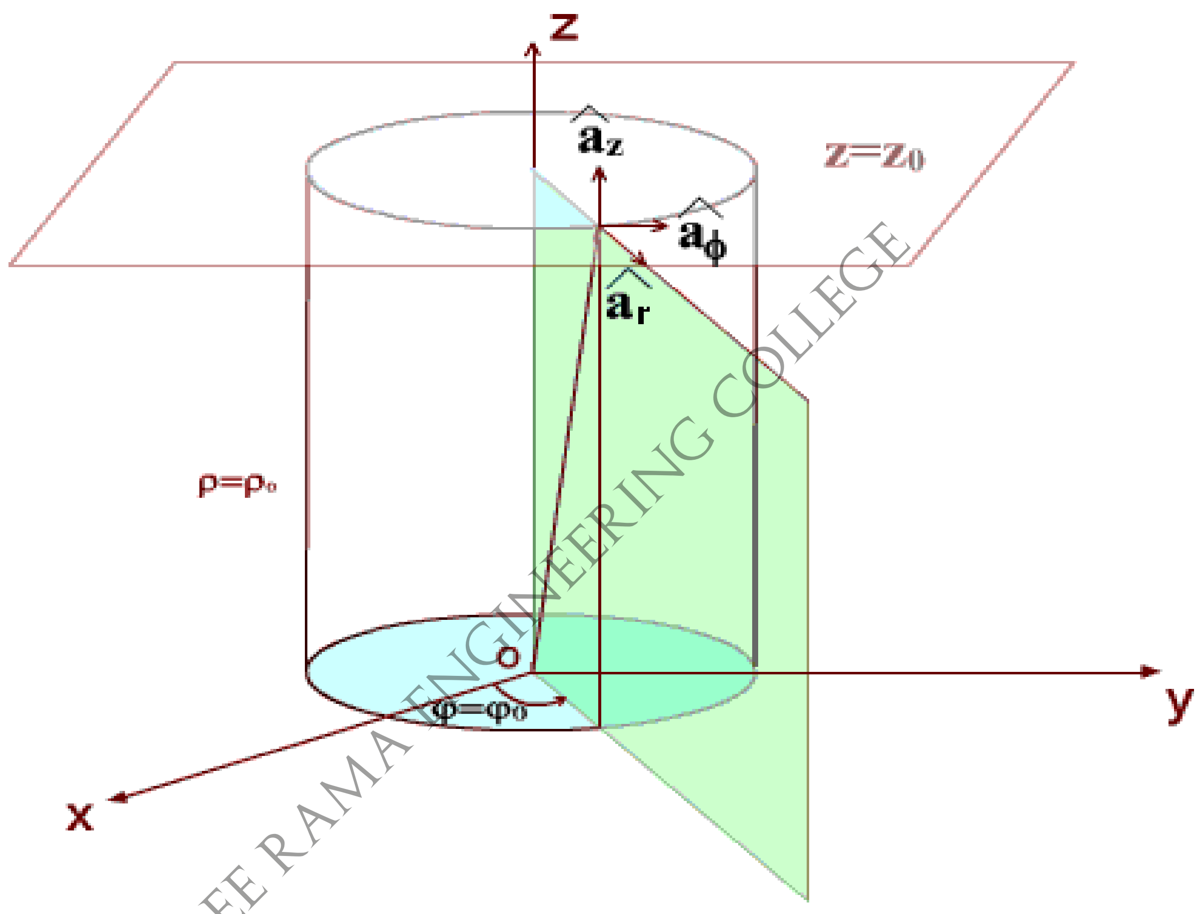
# CO-ORDINATE SYSTEM

- For cylindrical coordinate system we have $(u, v, w) = (\rho, \emptyset, z)$ a point $P(\rho_0, \emptyset_0, z_o)$ is determined as the point of intersection of

1) a cylindrical surface $\rho = \rho_0$

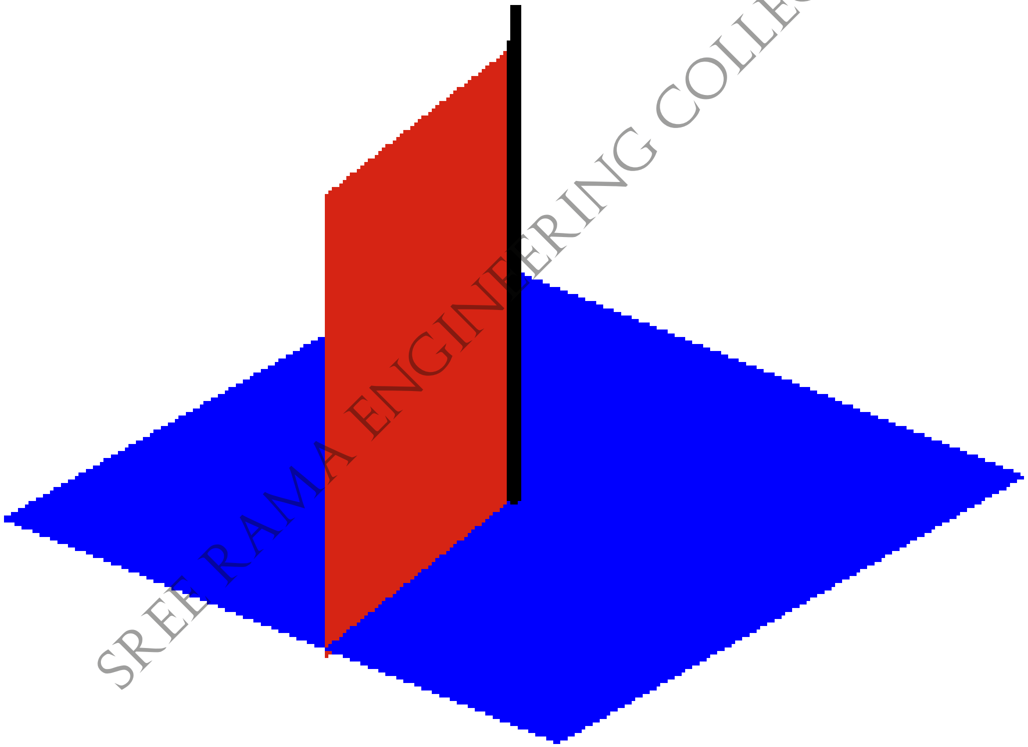
2) half plane containing the z-axis and making an angle with the xz plane $\emptyset = \emptyset_0$

3) a plane parallel to *xy* plane located at $z=z_o$ as shown in figure

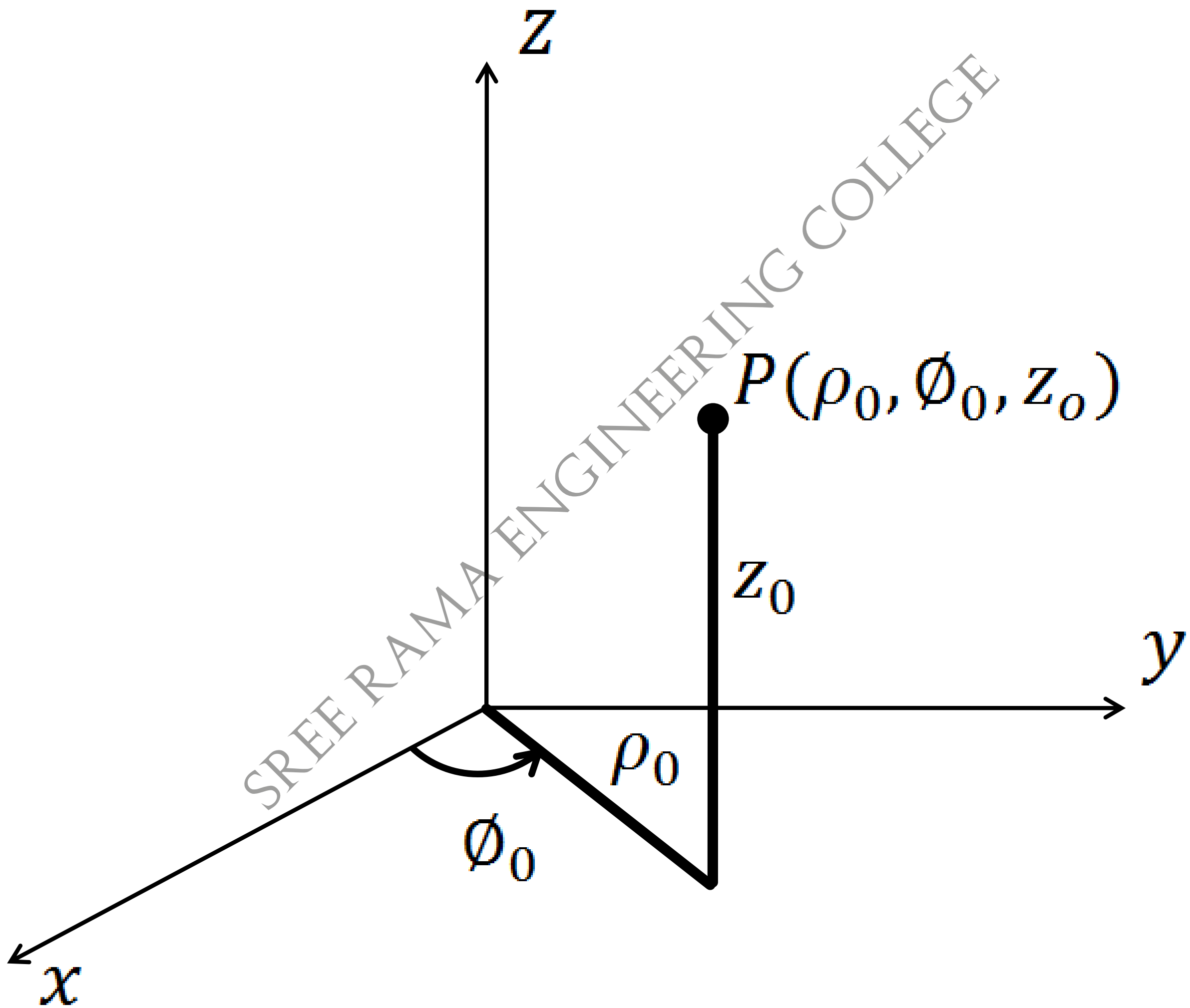


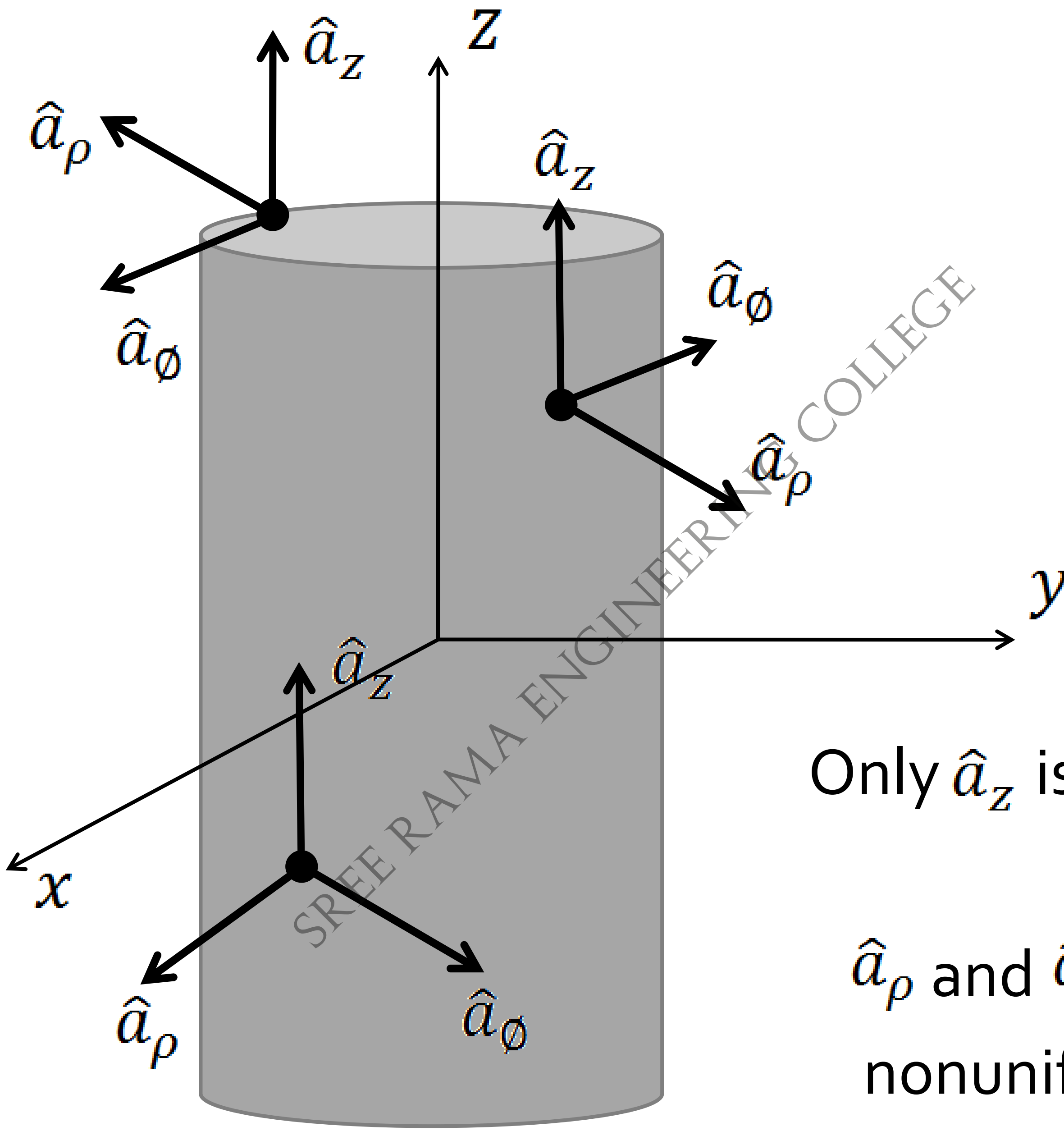


$$0 \leq \rho < \infty \qquad 0 \leq \emptyset < 2\pi \qquad -\infty < z < \infty$$



# A point in cylindrical coordinate system

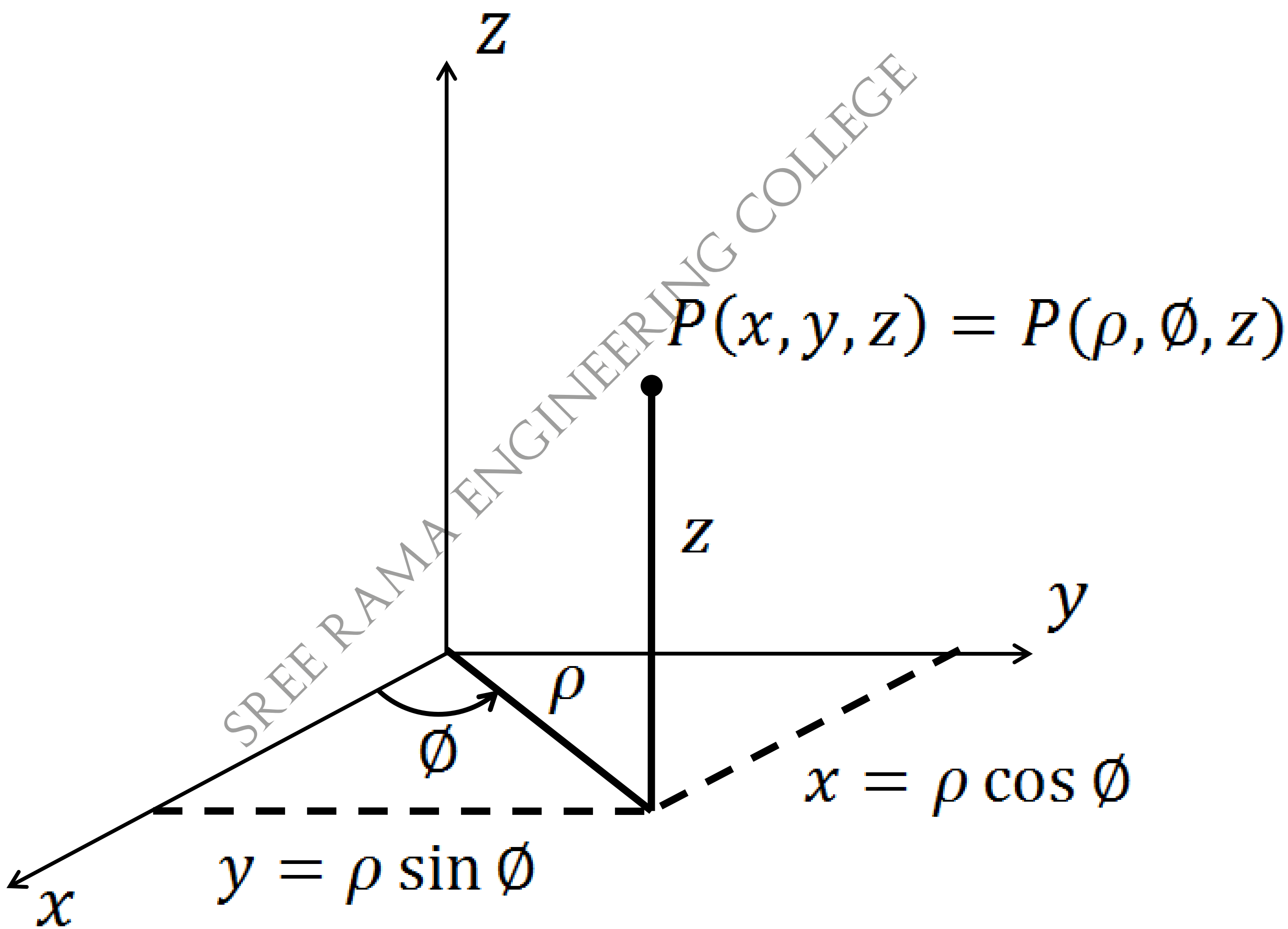





Only $\hat{a}_z$ is uniform.

$\hat{a}_\rho$ and $\hat{a}_\emptyset$ are nonuniform.

# Relationship between (x,y,z) and (ρ,ϕ,z)



$$x = \rho \cos \emptyset \qquad \rho = \sqrt{x^2 + y^2}$$

$$y = \rho \sin \emptyset \qquad ø = \tan^{-1} \frac{y}{x}$$

$$z = z \qquad z = z$$

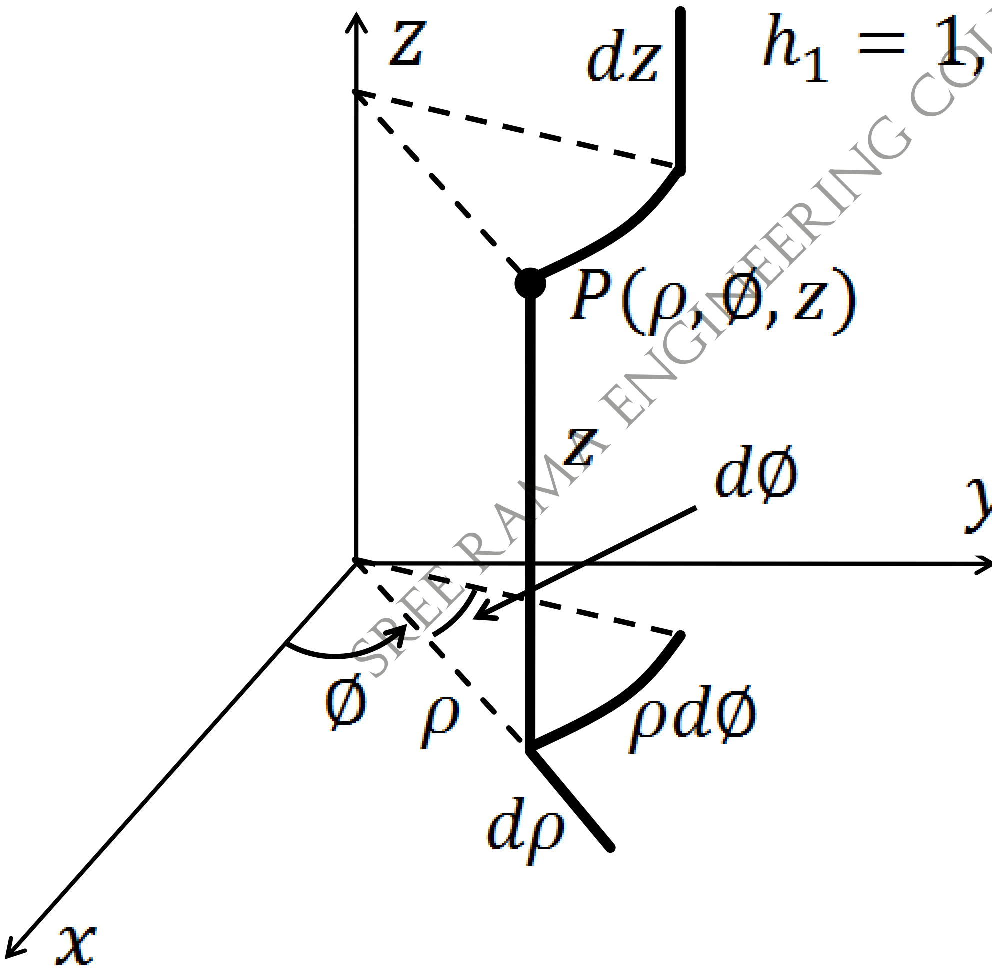
A vector can be written as

$$\vec{A} = A_\rho \hat{a}_\rho + A_ø \hat{a}_ø + A_z \hat{a}_z$$

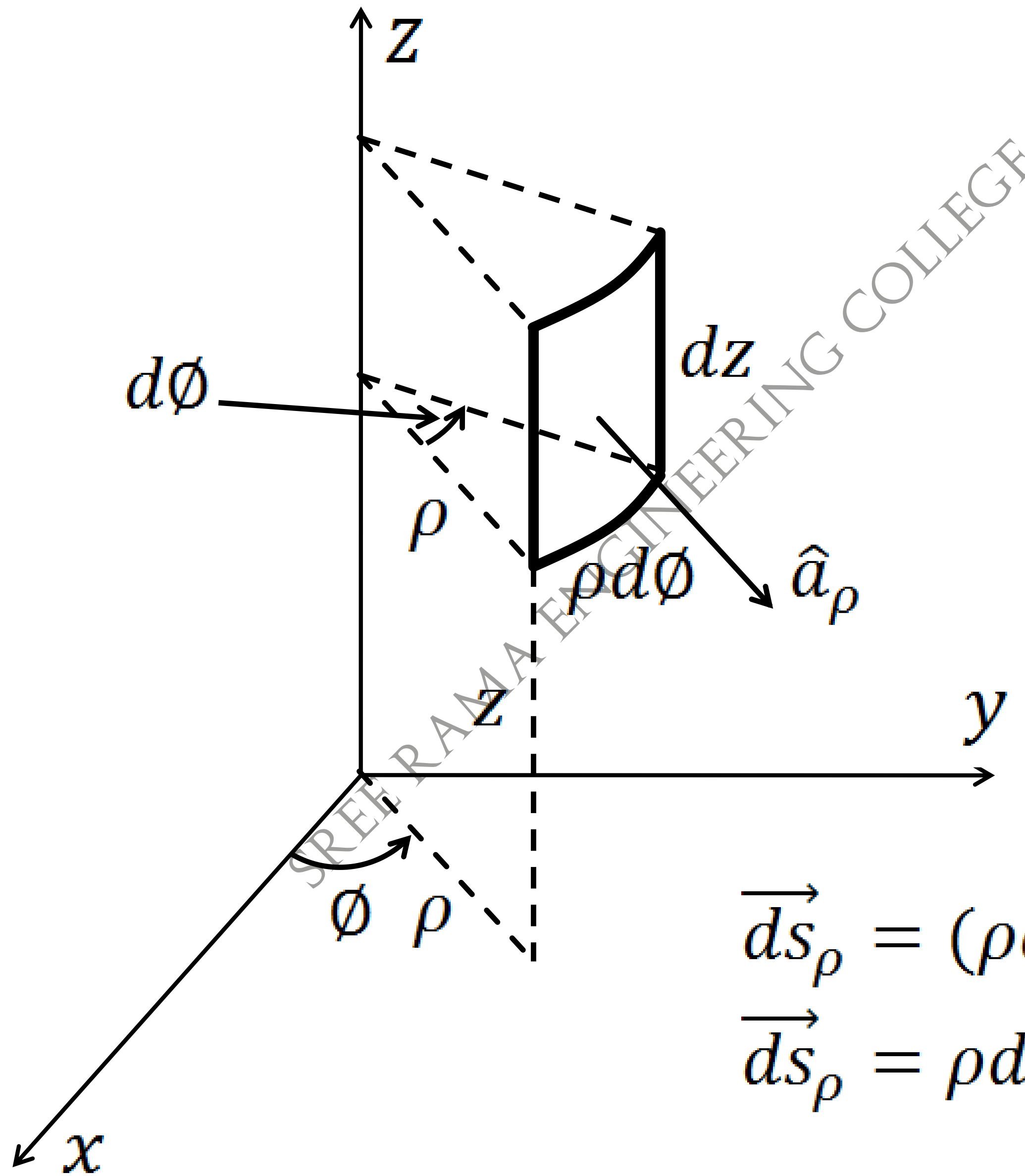
# Differential Length

$$\overrightarrow{dl} = d\rho\hat{a}_\rho + \rho d\phi\hat{a}_\phi + dz\hat{a}_z$$

$$h_1 = 1, h_2 = \rho, h_3 = 1$$

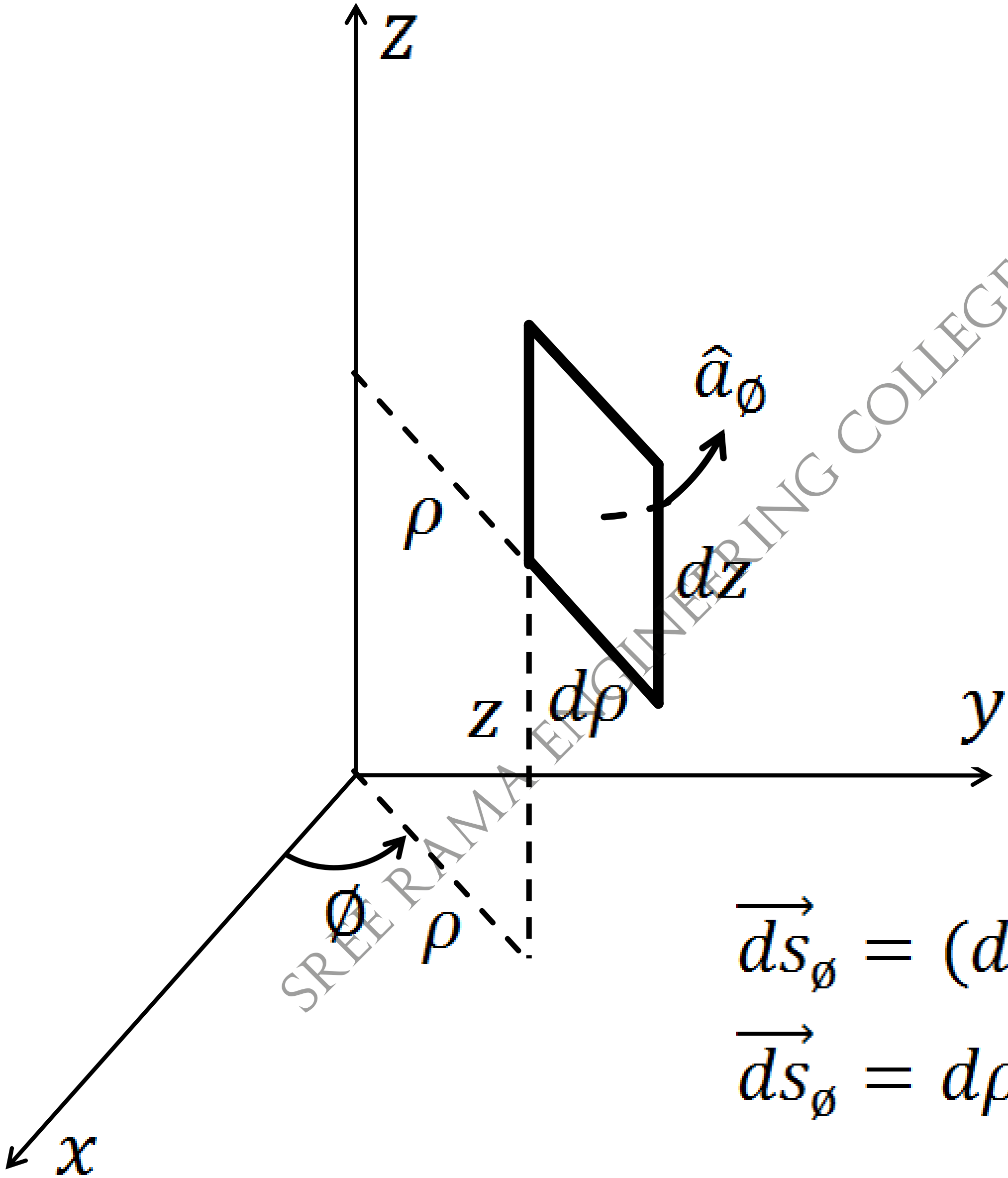


# Differential Areas



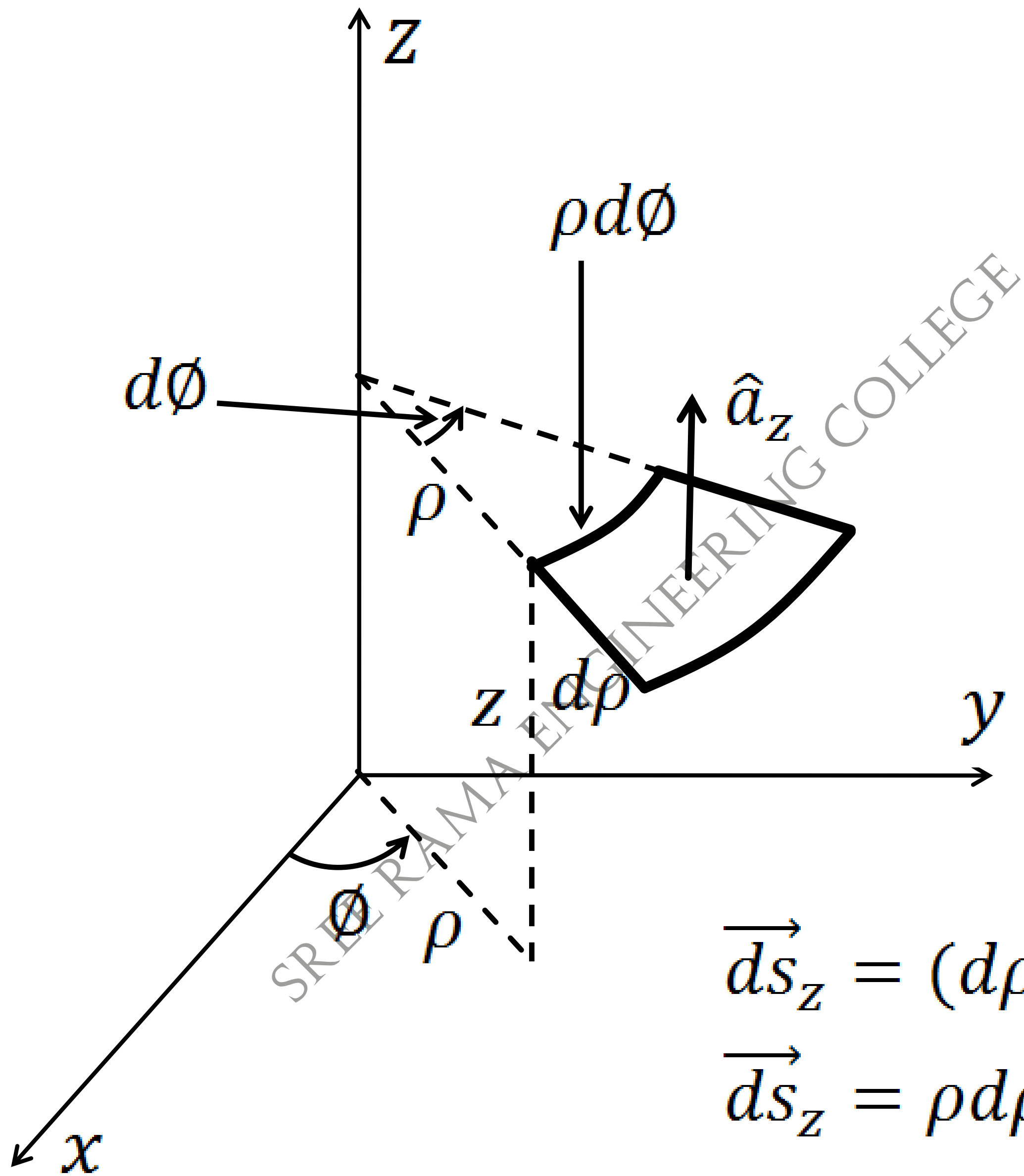

$$\overrightarrow{ds_\rho} = (\rho d\o \hat{a}_\o \times dz \hat{a}_z)$$

$$\overrightarrow{ds_\rho} = \rho d\o dz \hat{a}_\rho$$




$$\overrightarrow{ds_{ø}} = (dz\hat{a}_z \times d\rho\hat{a}_\rho)$$

$$\overrightarrow{ds_{ø}} = d\rho dz\hat{a}_z$$

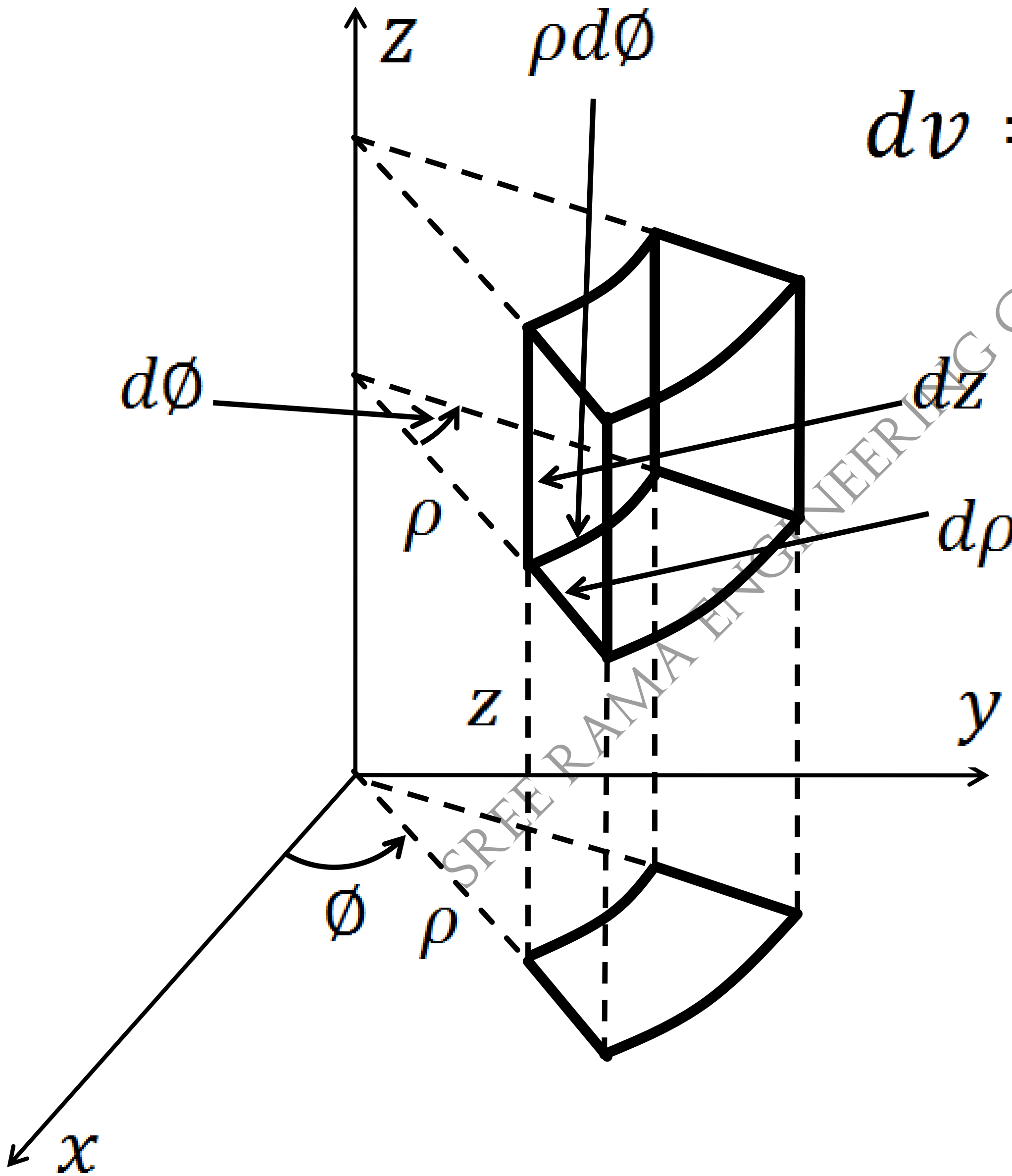





$$\overrightarrow{ds_z} = (d\rho\hat{a}_\rho \times \rho d\phi\hat{a}_\phi)$$

$$\overrightarrow{ds_z} = \rho d\rho d\phi\hat{a}_z$$

# Differential Volume




$$dv = (d\rho)(\rho d\emptyset)(dz)$$

$$dv = \rho d\rho d\emptyset \, dz$$

## Transformation between Cartesian and Cylindrical coordinates:

Let us consider

$$\vec{A} = A_\rho \hat{a}_\rho + A_ø \hat{a}_ø + A_z \hat{a}_z$$

is to be expressed in Cartesian co-ordinates as

$$\vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$$

In doing so we note that

$$A_x = \vec{A} \cdot \hat{a}_x = (A_\rho \hat{a}_\rho + A_ø \hat{a}_ø + A_z \hat{a}_z) \cdot \hat{a}_x$$

$$A_y = \vec{A} \cdot \hat{a}_y = (A_\rho \hat{a}_\rho + A_ø \hat{a}_ø + A_z \hat{a}_z) \cdot \hat{a}_y$$

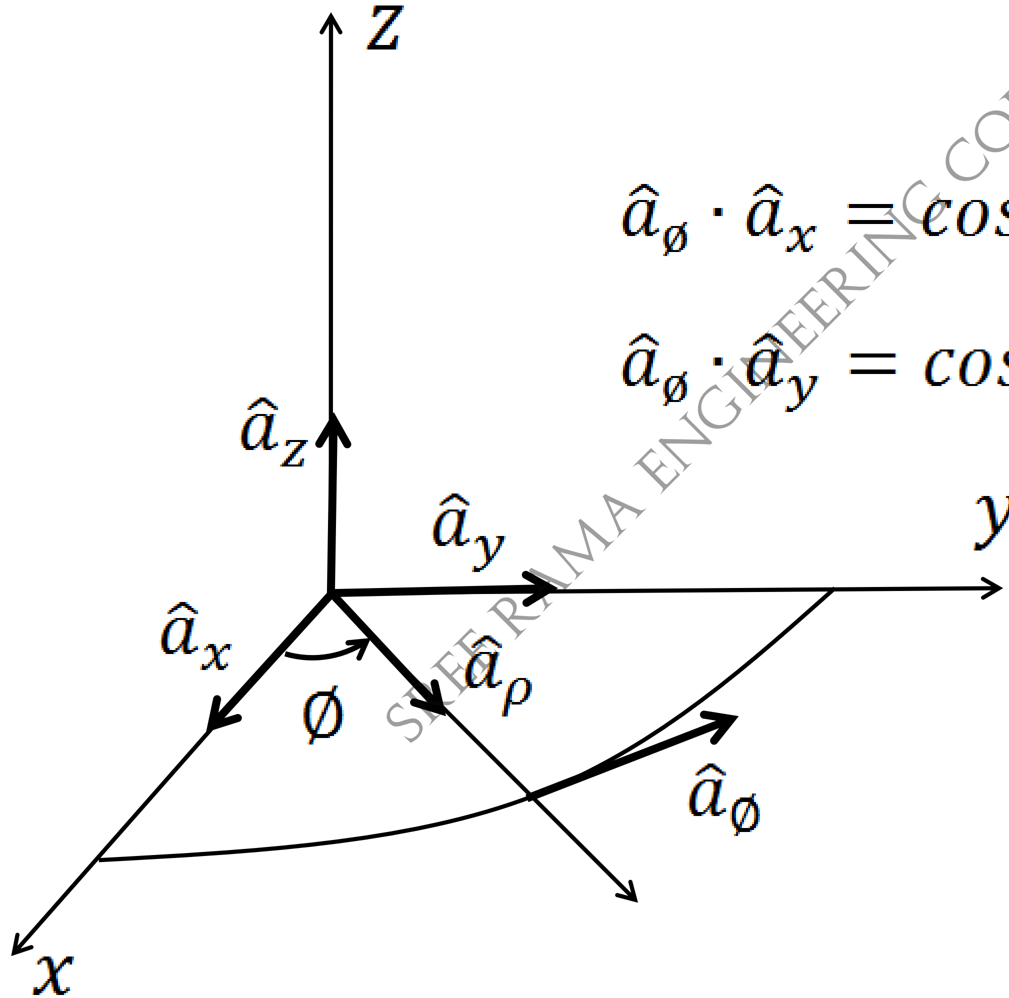
$$A_z = \vec{A} \cdot \hat{a}_z = (A_\rho \hat{a}_\rho + A_ø \hat{a}_ø + A_z \hat{a}_z) \cdot \hat{a}_z$$

$$\hat{a}_\rho \cdot \hat{a}_x = \cos ø$$

$$\hat{a}_\rho \cdot \hat{a}_y = \cos(90 - \emptyset) = \sin ø$$

$$\hat{a}_ø \cdot \hat{a}_x = \cos\left(\emptyset + \frac{\pi}{2}\right) = -\sin ø$$

$$\hat{a}_ø \cdot \hat{a}_y = \cos ø$$



Therefore we can write

$$A_x = \vec{A} \cdot \hat{a}_x = A_\rho \cos \emptyset - A_\emptyset \sin \emptyset$$

$$A_y = \vec{A} \cdot \hat{a}_y = A_\rho \sin \emptyset + A_\emptyset \cos \emptyset$$

$$A_z = \vec{A} \cdot \hat{a}_z = A_z$$

These relations can be put conveniently in the matrix form as

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

# SPHERICAL

# CO-ORDINATE SYSTEM

For spherical polar coordinate system we have

$$(u, v, w) = (r, \theta, \emptyset)$$

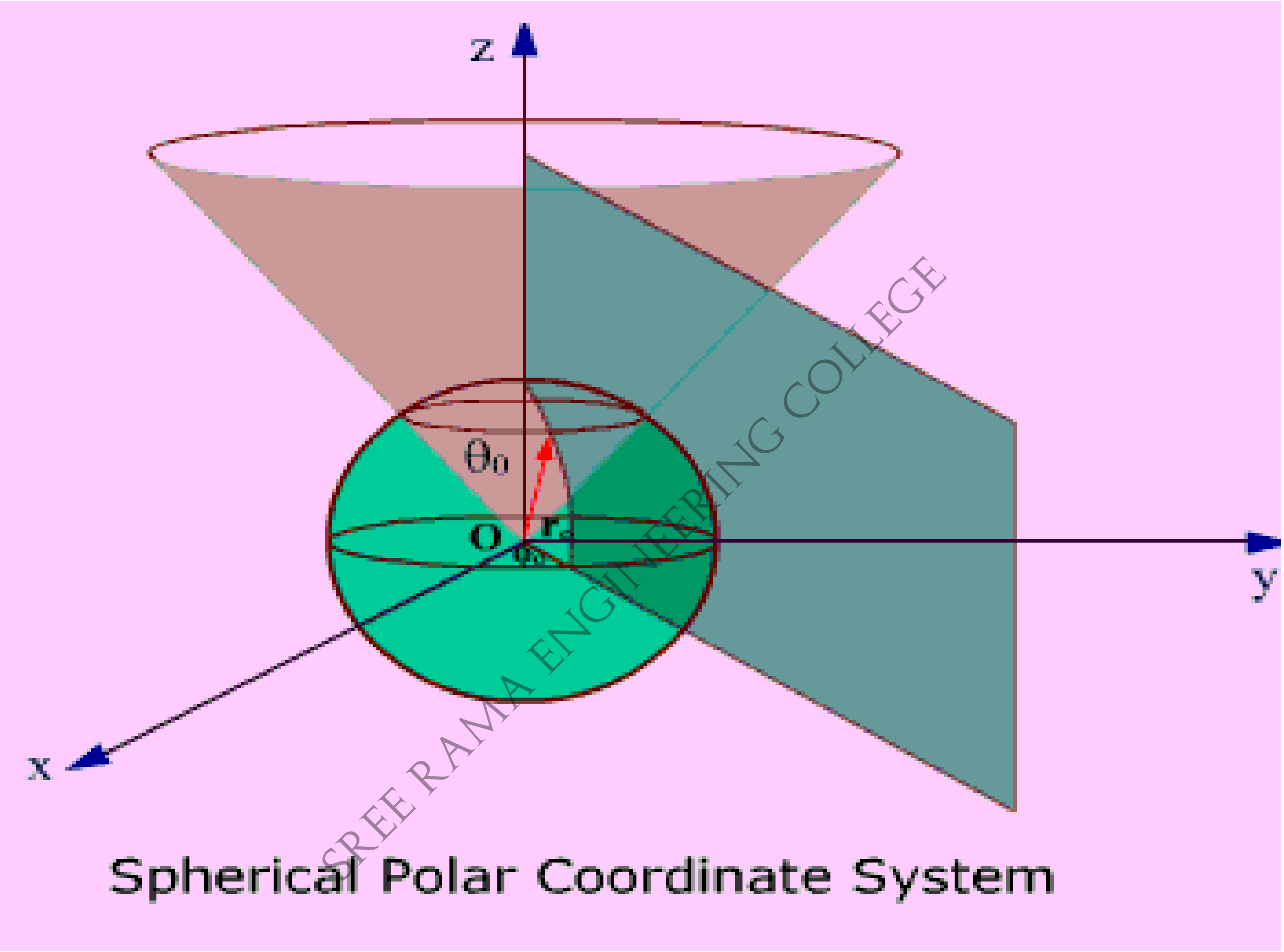
A point $P(r_0, \theta_0, \emptyset_o)$

is represented as the intersection of

(i) Spherical surface $r = r_0$

(ii) Conical surface $\theta = \theta_0$

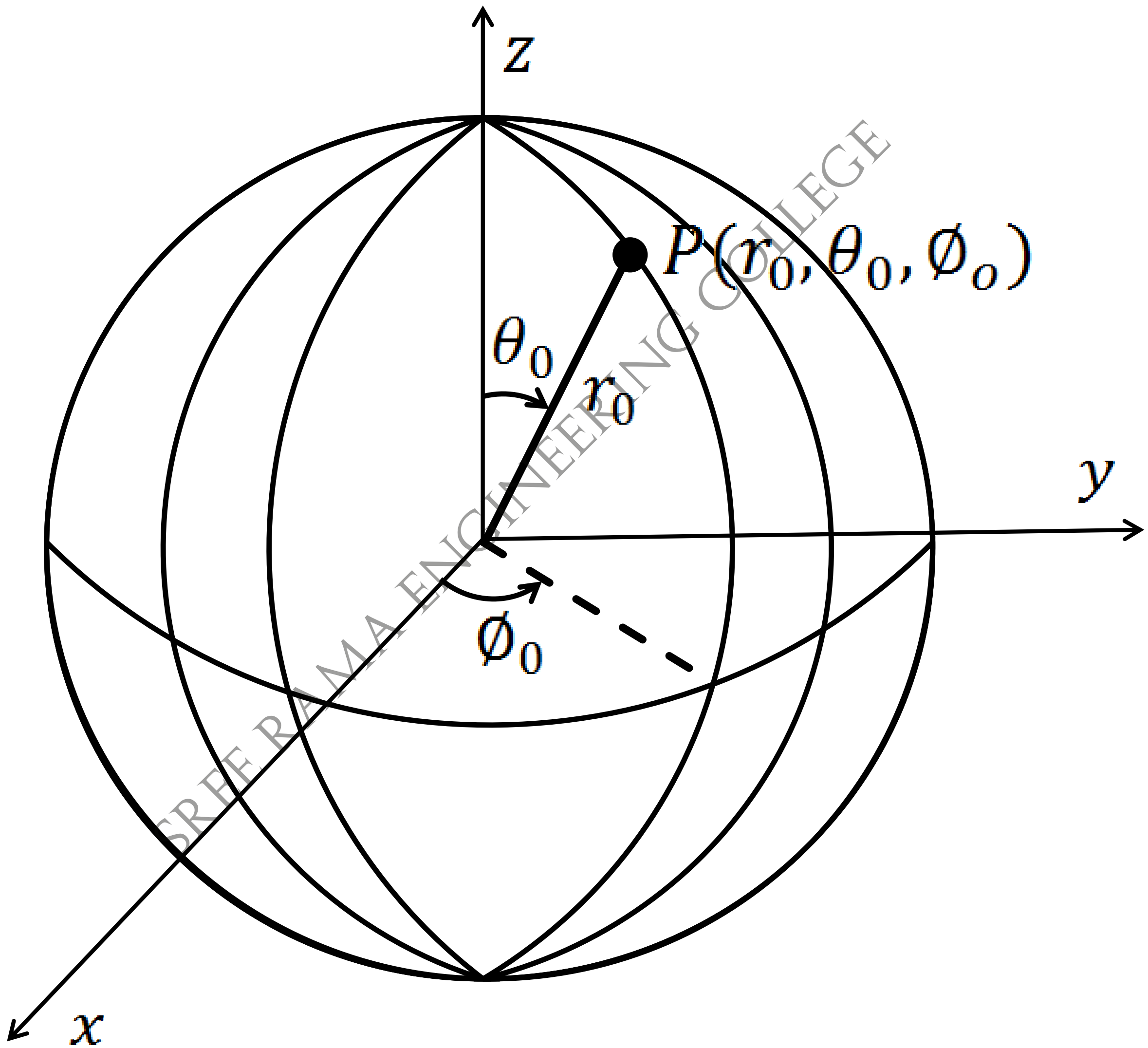
(iii) half plane containing z-axis making angle with the *xz* plane $\emptyset = \emptyset_0$ as shown in the figure

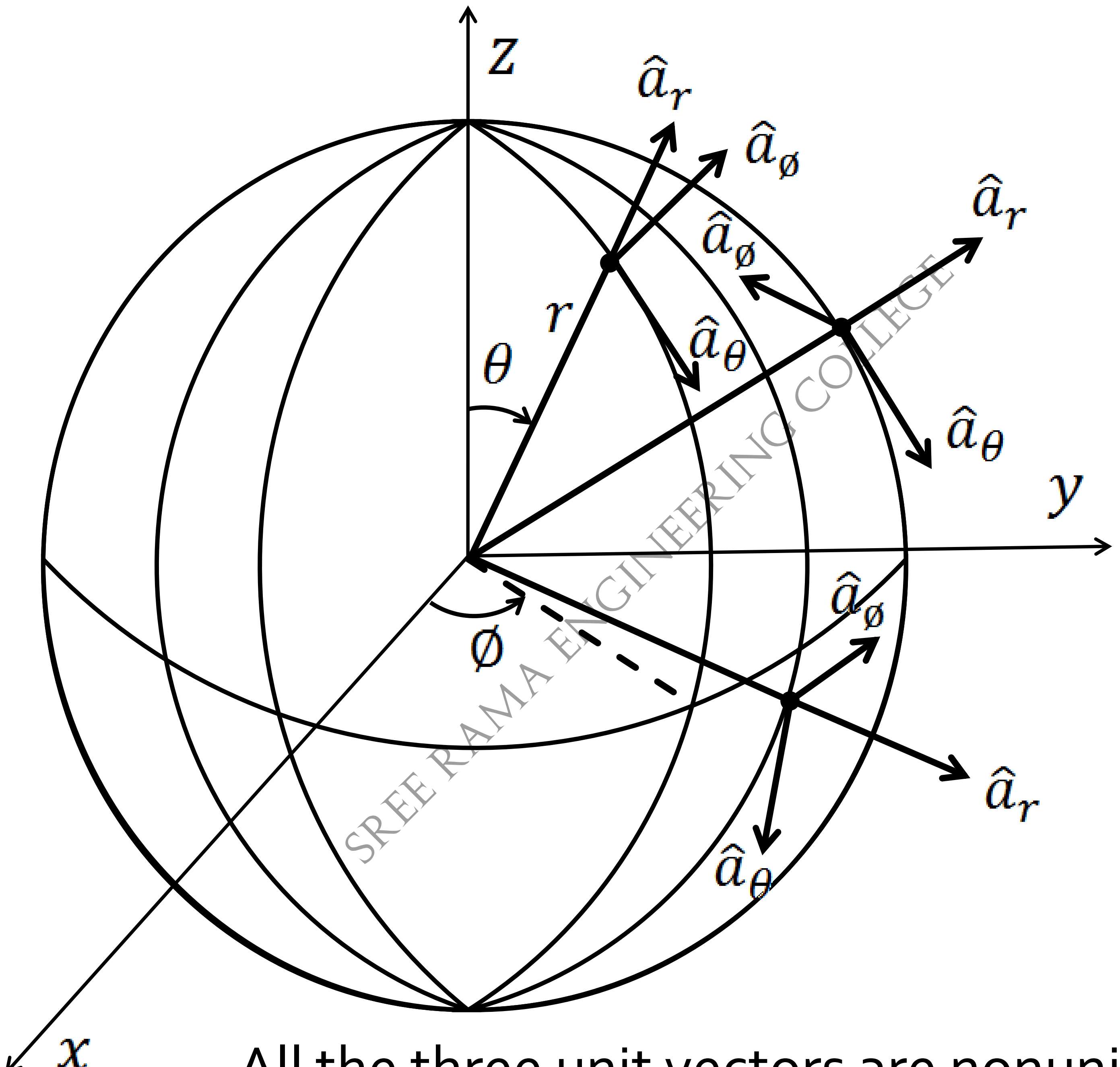




$$0 \le r < \infty \qquad 0 \le \theta \le \pi \qquad 0 \le \emptyset < 2\pi$$

# A point in spherical coordinate system







All the three unit vectors are nonuniform

The unit vectors satisfy the following relationships

$$\hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi$$

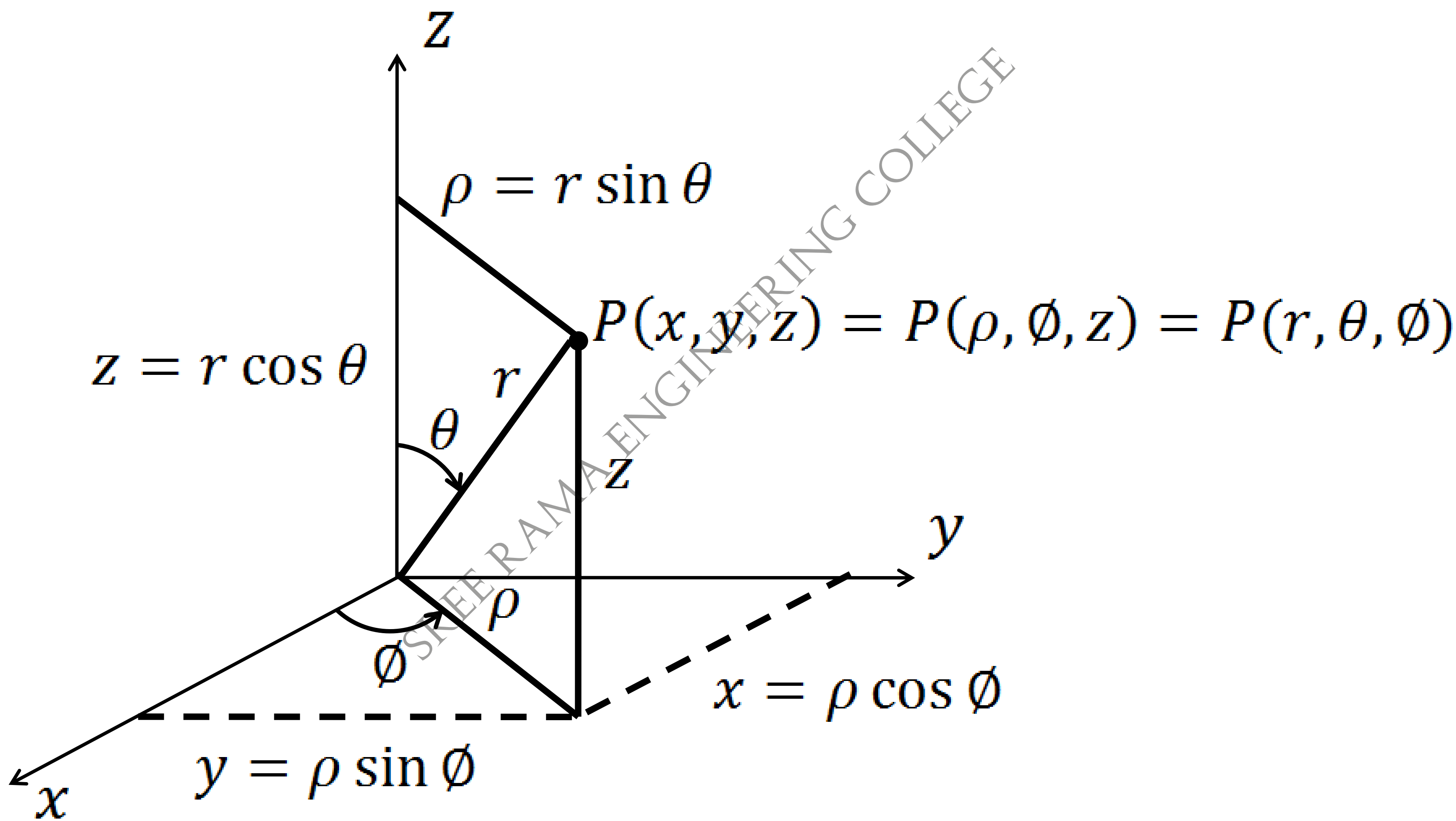
$$\hat{a}_\theta \times \hat{a}_\phi = \hat{a}_r$$

$$\hat{a}_\phi \times \hat{a}_r = \hat{a}_\theta$$

$$\hat{a}_r \cdot \hat{a}_r = \hat{a}_\theta \cdot \hat{a}_\theta = \hat{a}_\phi \cdot \hat{a}_\phi = 1$$

$$\hat{a}_r \cdot \hat{a}_\theta = \hat{a}_\theta \cdot \hat{a}_\phi = \hat{a}_\phi \cdot \hat{a}_r = 0$$

# Relationships between space variables (x,y,z), (ρ,ϕ,z) and (r,θ,ϕ)



$$x = \rho \cos \emptyset$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$y = \rho \sin \emptyset$$

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}$$

$$z = r \cos \theta$$



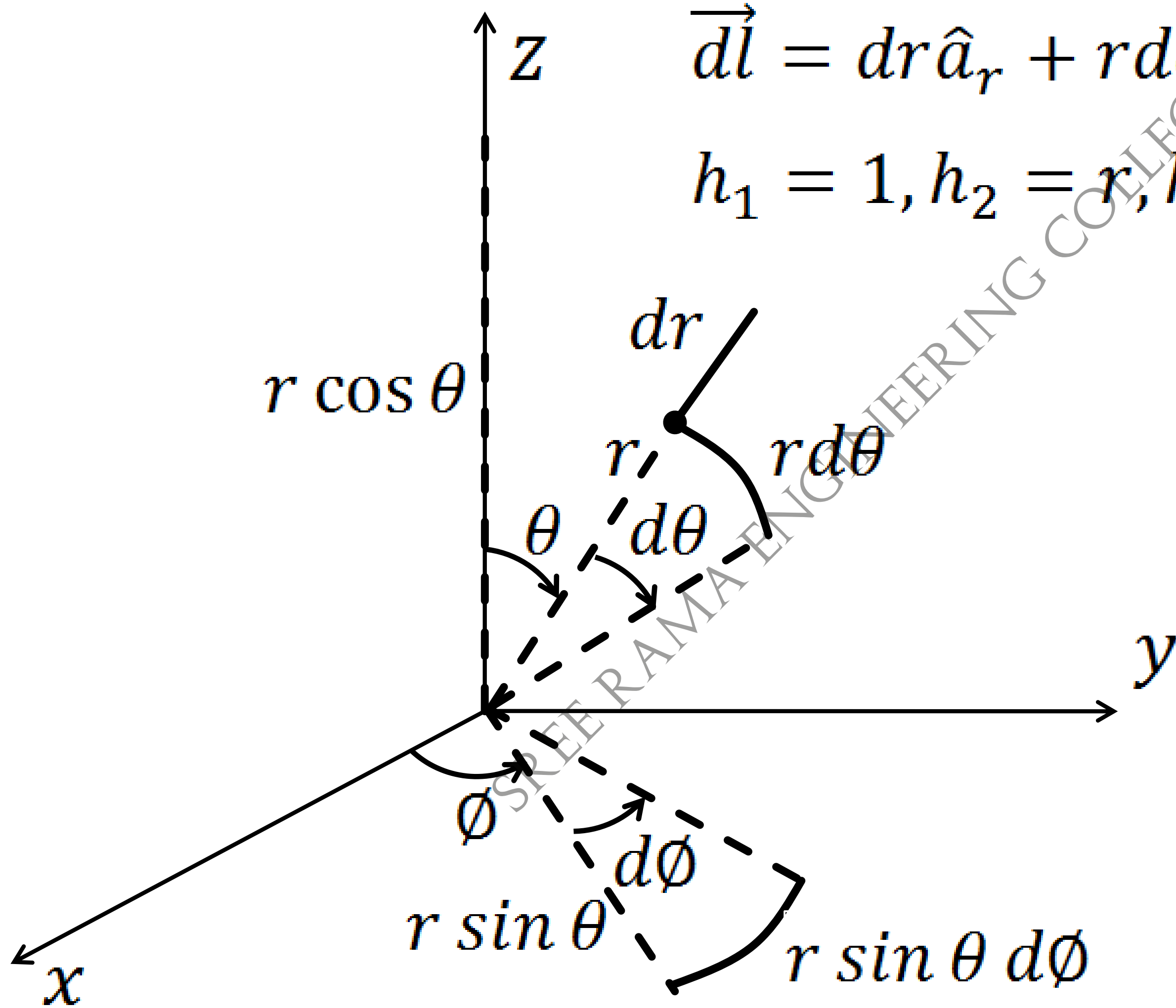
A vector in spherical polar co-ordinates is written as

$$\vec{A} = A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\emptyset \hat{a}_\emptyset$$

# Differential Length

$$\overrightarrow{dl} = dr\hat{a}_r + rd\theta\hat{a}_\theta + r\sin\theta\, d\phi\hat{a}_\phi$$

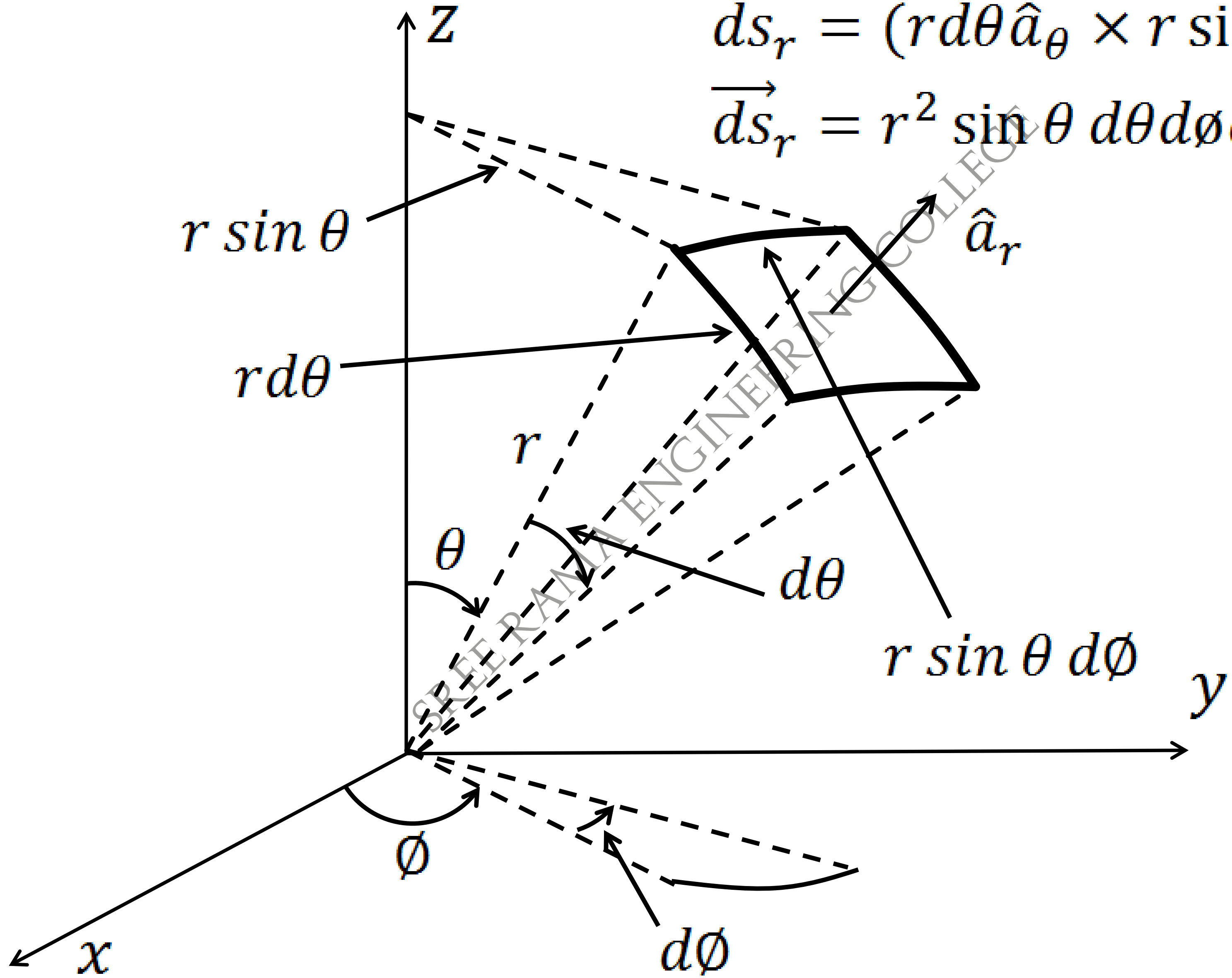
$$h_1 = 1, h_2 = r, h_3 = r\sin\theta$$



# Differential Areas

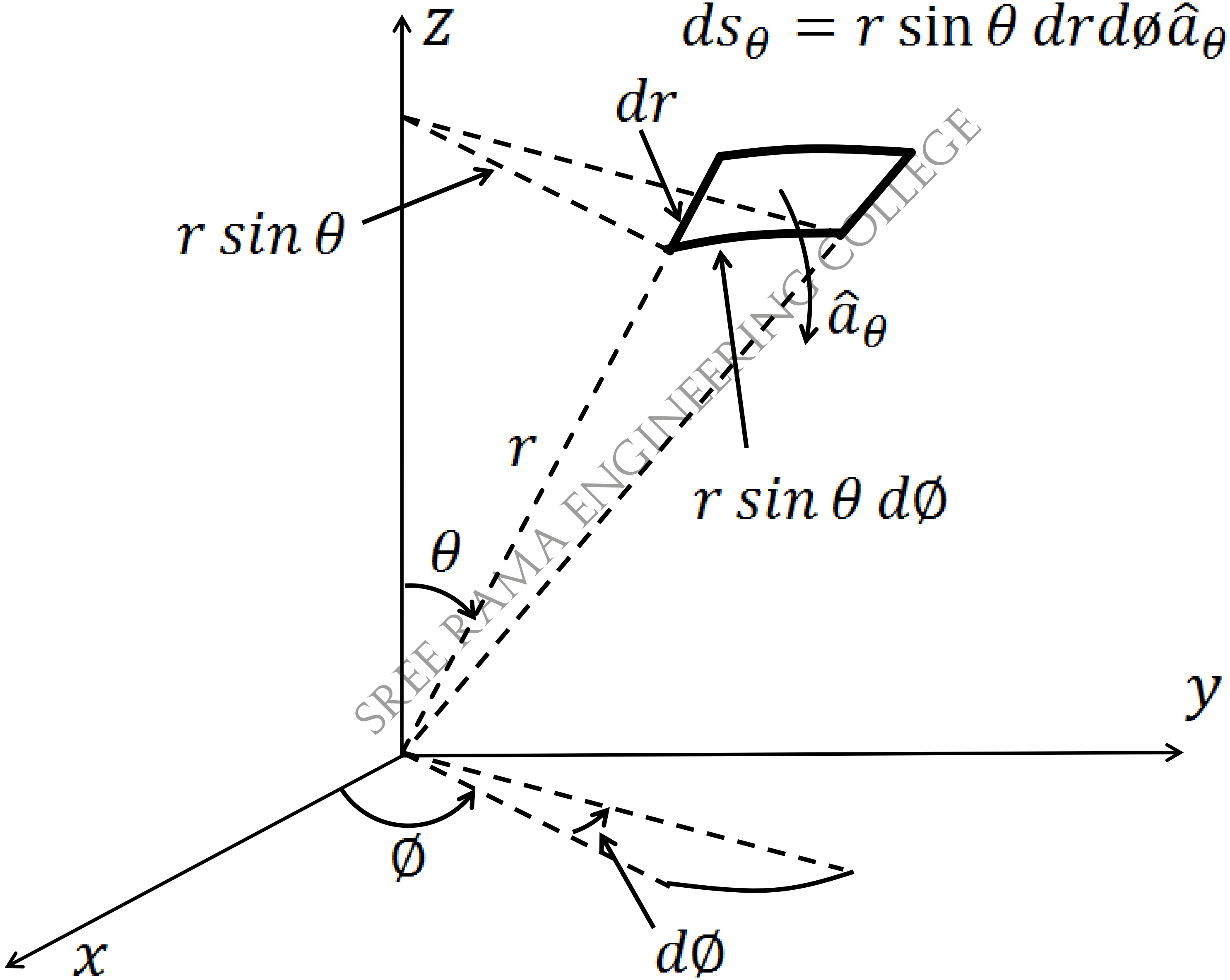
$$\overrightarrow{ds_r} = (rd\theta\,\hat{a}_\theta \times r\sin\theta\, d\o\,\hat{a}_\o)$$

$$\overrightarrow{ds_r} = r^2 \sin\theta\, d\theta d\o\,\hat{a}_r$$



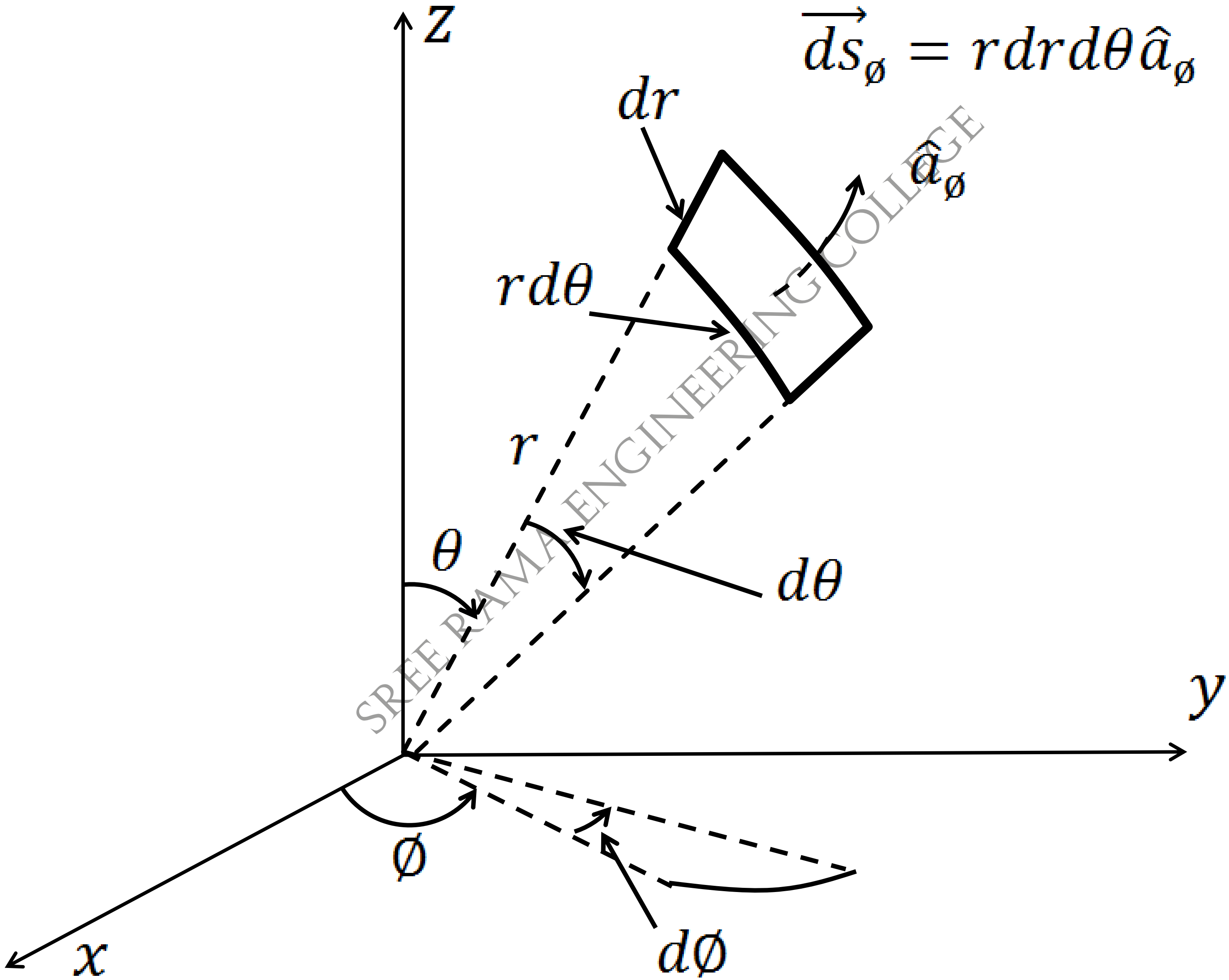
$$\overrightarrow{ds_\theta} = (r \sin\theta \, d\phi \hat{a}_\phi \times dr \hat{a}_r)$$

$$\overrightarrow{ds_\theta} = r \sin\theta \, dr d\phi \hat{a}_\theta$$



$$\overrightarrow{ds_\phi} = (dr\hat{a}_r \times rd\theta\hat{a}_\theta)$$

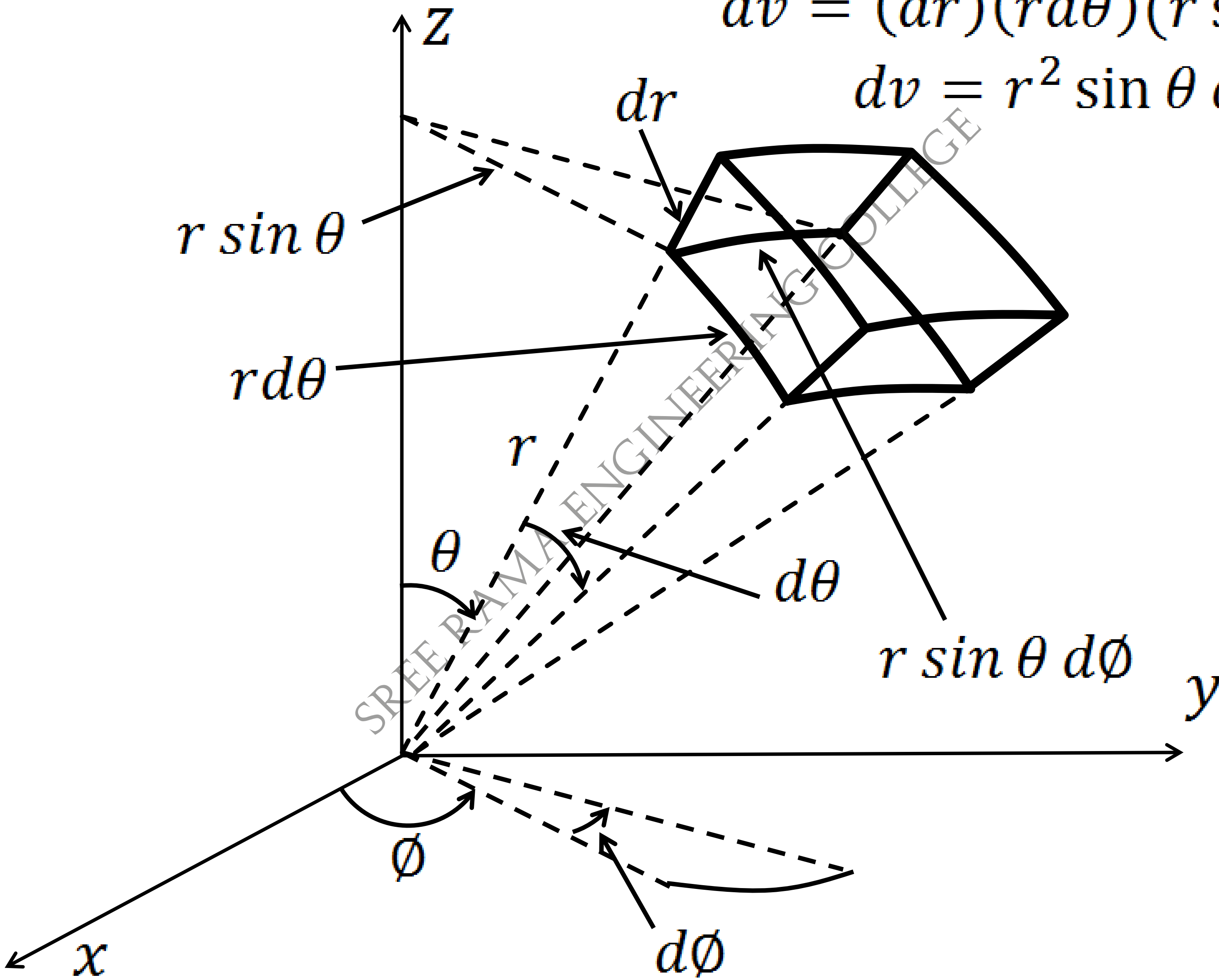
$$\overrightarrow{ds_\phi} = rdrd\theta\hat{a}_\phi$$



# Differential Volume

$$dv = (dr)(rd\theta)(r\sin\theta\, d\phi)$$

$$dv = r^2 \sin\theta\, dr d\theta d\phi$$



# Transformation between Cartesian and Spherical coordinates:

Given a vector

$$\vec{A} = A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\emptyset \hat{a}_\emptyset$$

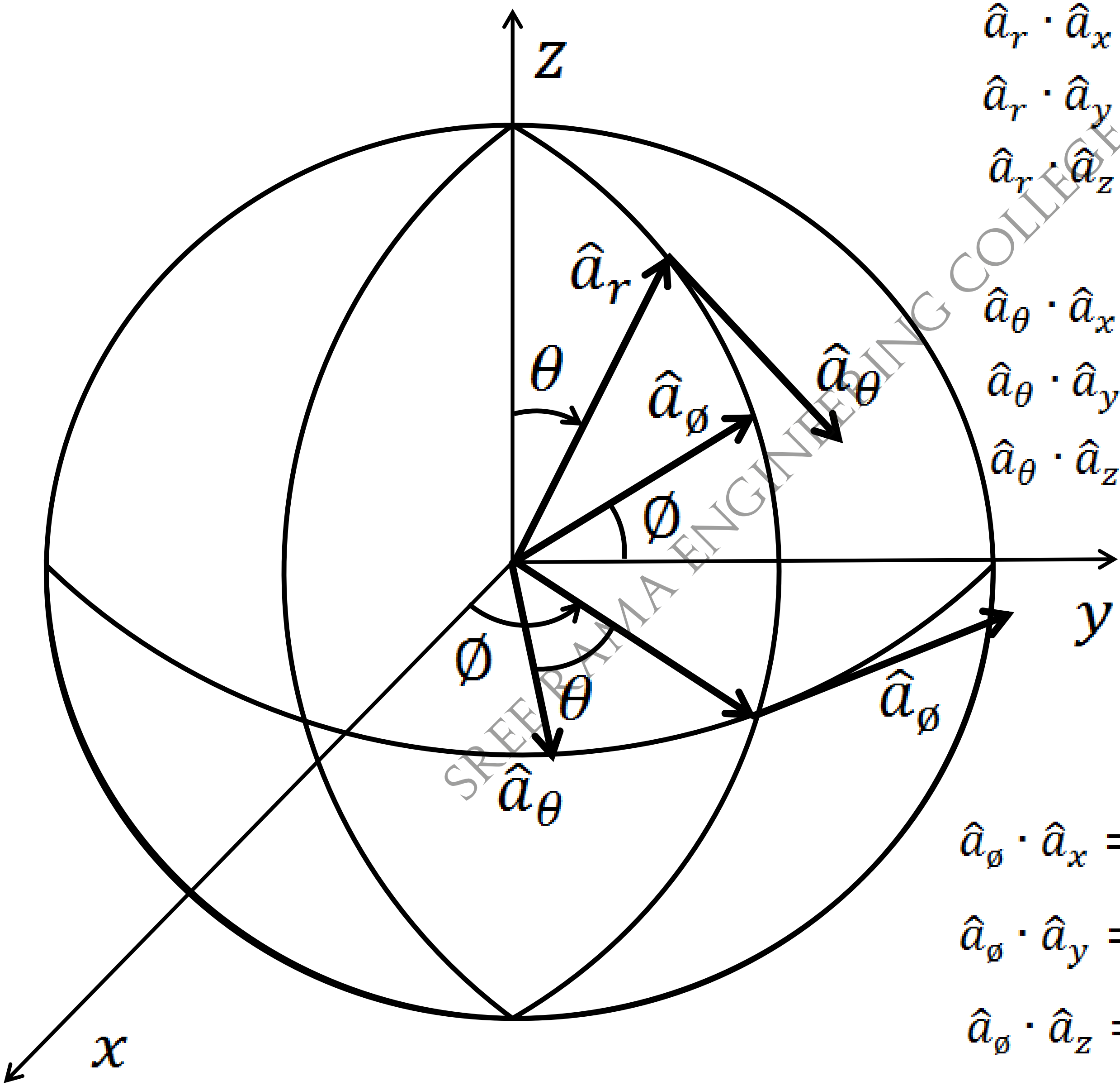
in the spherical polar coordinate system, its components in the Cartesian coordinate system can be found out as follows:

$$A_x = \vec{A} \cdot \hat{a}_x = (A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\emptyset \hat{a}_\emptyset) \cdot \hat{a}_x$$

$$A_y = \vec{A} \cdot \hat{a}_y = (A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\emptyset \hat{a}_\emptyset) \cdot \hat{a}_y$$

$$A_z = \vec{A} \cdot \hat{a}_z = (A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\emptyset \hat{a}_\emptyset) \cdot \hat{a}_z$$

# Coordinate transformation between rectangular and spherical




$$\hat{a}_r \cdot \hat{a}_x = \sin\theta\cos ø$$
$$\hat{a}_r \cdot \hat{a}_y = \sin\theta\sin ø$$
$$\hat{a}_r \cdot \hat{a}_z = \cos\theta$$

$$\hat{a}_\theta \cdot \hat{a}_x = \cos\theta\cos ø$$
$$\hat{a}_\theta \cdot \hat{a}_y = \cos\theta\sin ø$$
$$\hat{a}_\theta \cdot \hat{a}_z = \cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$$

$$\hat{a}_ø \cdot \hat{a}_x = \cos\left(Ø + \frac{\pi}{2}\right) = -\sin ø$$
$$\hat{a}_ø \cdot \hat{a}_y = \cos ø$$
$$\hat{a}_ø \cdot \hat{a}_z = 0$$

$$A_x = \vec{A} \cdot \hat{a}_x = A_r \sin\theta \cos\emptyset + A_\theta \cos\theta \cos\emptyset - A_\emptyset \sin\emptyset$$

$$A_y = \vec{A} \cdot \hat{a}_y = A_r \sin\theta \sin\emptyset + A_\theta \cos\theta \sin\emptyset + A_\emptyset \cos\emptyset$$

$$A_z = \vec{A} \cdot \hat{a}_z = A_r \cos\theta - A_\theta \sin\theta$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\emptyset & \cos\theta\cos\emptyset & -\sin\emptyset \\ \sin\theta\sin\emptyset & \cos\theta\sin\emptyset & \cos\emptyset \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\emptyset \end{bmatrix}$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\emptyset \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\emptyset & \sin\theta\sin\emptyset & \cos\theta \\ \cos\theta\cos\emptyset & \cos\theta\sin\emptyset & -\sin\theta \\ -\sin\emptyset & \cos\emptyset & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

# ELECTROMAGNETIC FIELD THEORY
## (20A02403T)

## VECTOR ALGEBRA



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## 1.3 Representation of a Vector

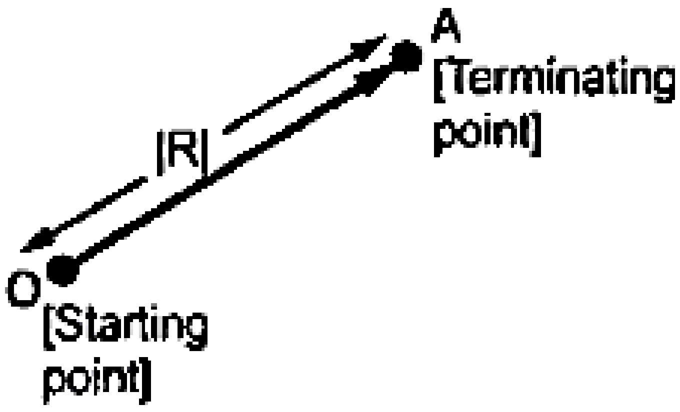



**Fig. 1.1 Representation of a vector**

In two dimensions, a vector can be represented by a straight line with an arrow in a plane. This is shown in the Fig. 1.1. The length of the segment is the magnitude of a vector while the arrow indicates the direction of the vector in a given co-ordinate system. The vector shown in the Fig. 1.1 is symbolically denoted as $\overline{\mathbf{OA}}$. The point O is its starting point while A is its terminating point. Its length is called its magnitude, which is R for the vector OA shown. It is represented as $\left|\overline{OA}\right| = R$. It is the distance between the starting point and terminating point of a vector.

**Key Point:** *The vector hereafter will be indicated by bold letter with a bar over it.*

## 1.4 Vector Algebra

1. Scaling 2. Addition 3. Subtraction

### 1.4.1 Scaling of Vector

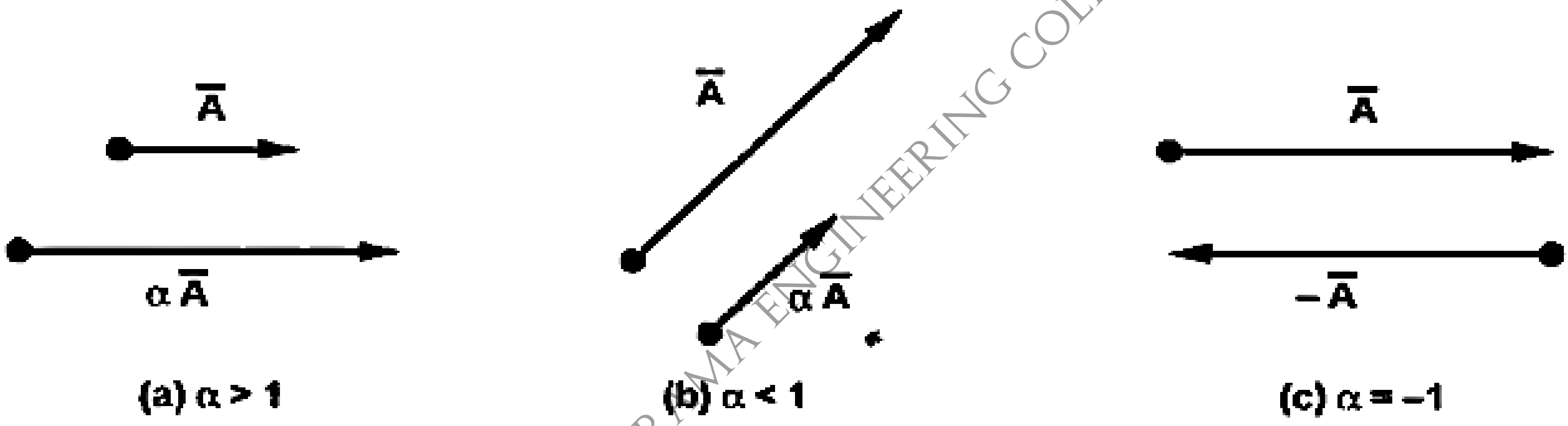



**Fig. 1.3 Multiplication by a scalar**

**Key Point:** *Thus if $\alpha$ is negative. the magnitude of vector changes by $\alpha$ times while the direction becomes exactly opposite to the original vector, after multiplication.*

## 1.4.2 Addition of Vectors

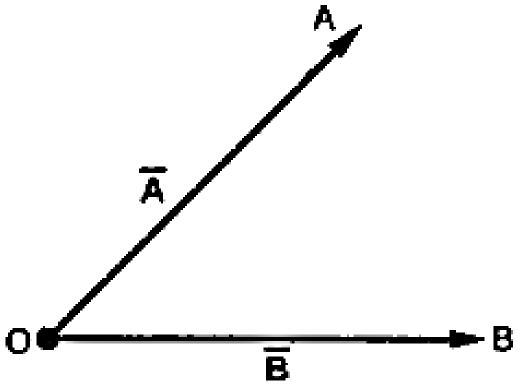



Fig. 1.4 Coplanar vectors

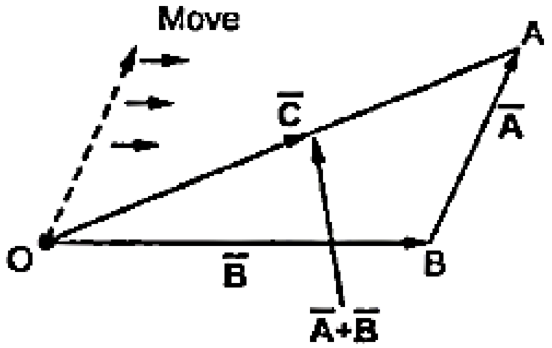



Fig. 1.5 Addition of vectors

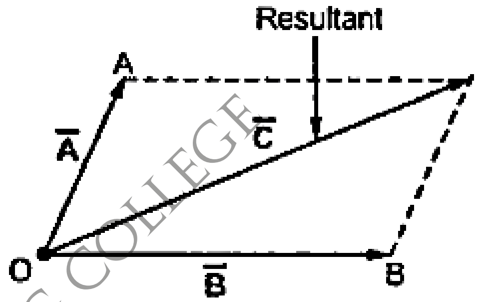
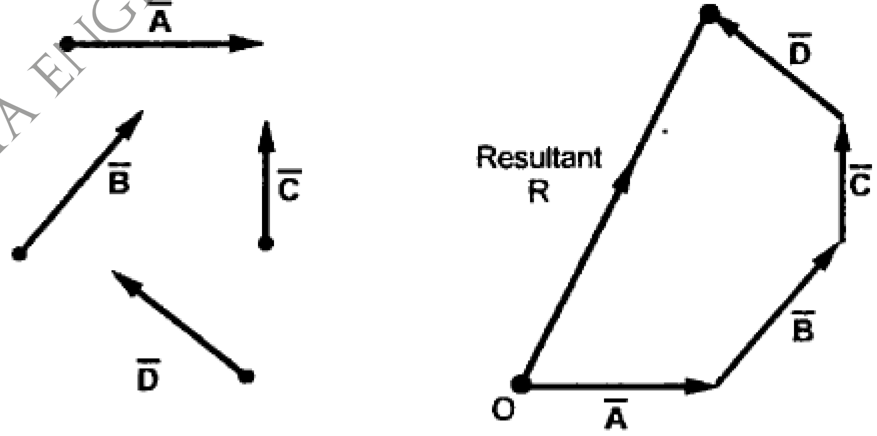



Fig. 1.6 Parallelogram rule for addition




(a) Four vectors (b) Sum of the four vectors

$$\overline{R} = \overline{A} + \overline{B} + \overline{C} + \overline{D}$$

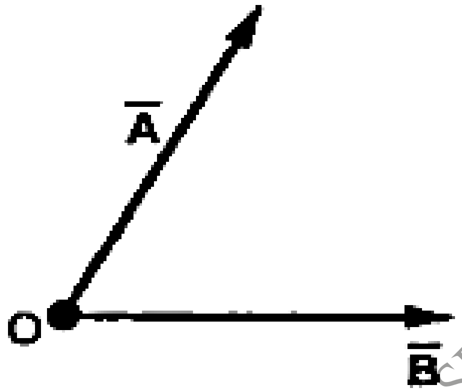
Fig. 1.7

### 1.4.3 Subtraction of Vectors

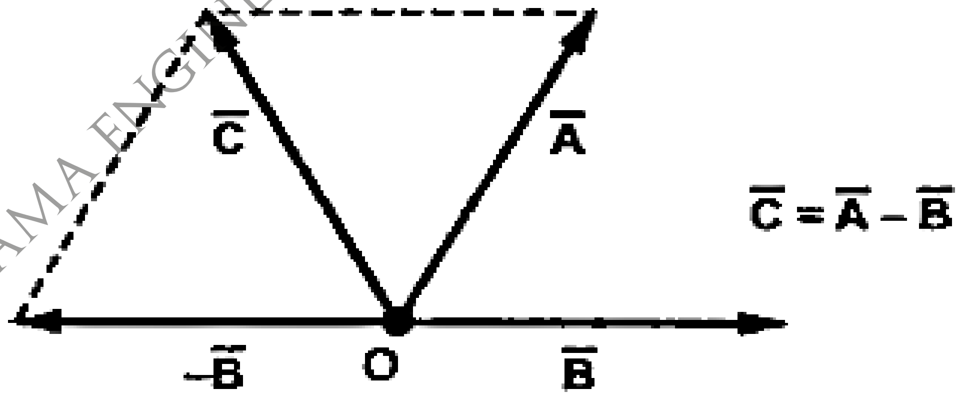
The subtraction of vectors can be obtained from the rules of addition. If $\overline{B}$ is to be subtracted from $\overline{A}$ then based on addition it can be represented as,

$$\overline{C} = \overline{A} + (-\overline{B})$$

Thus reverse the sign of $\overline{B}$ i.e. reverse its direction by multiplying it with $-1$ and then add it to $\overline{A}$ to obtain the subtraction. This is shown in the Fig. 1.8 (a) and (b).




(a) Vectors




(b) Subtraction of vectors

Fig. 1.8

### 1.3.1 Unit Vector

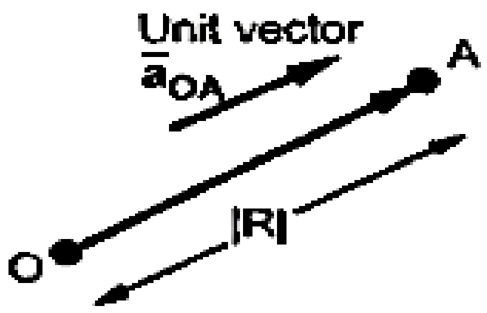



Fig. 1.2 Unit vector

A unit vector has a function to indicate the direction. Its magnitude is always unity, irrespective of the direction which it indicates and the co-ordinate system under consideration. Thus for any vector, to indicate its direction a unit vector can be used. Consider a unit vector $\bar{a}_{OA}$ in the direction of $\overline{OA}$ as shown in the Fig. 1.2. This vector indicates the direction of $\overline{OA}$ but its magnitude is unity.

So vector $\overline{OA}$ can be represented completely as its magnitude R and the direction as indicated by unit vector along its direction.

$$\therefore \qquad \boxed{\overline{OA} = |\overline{OA}|\, \bar{a}_{OA} = R\, \bar{a}_{OA}}$$

where $\bar{a}_{OA}$ = Unit vector along the direction OA and $|\bar{a}_{OA}| = 1$

$$\boxed{\text{Unit vector } \bar{a}_{OA} = \frac{\overline{OA}}{|\overline{OA}|}}$$

### 1.6.2 Base Vectors

The base vectors are the unit vectors which are strictly oriented along the directions of the co-ordinate axes of the given co-ordinate system.

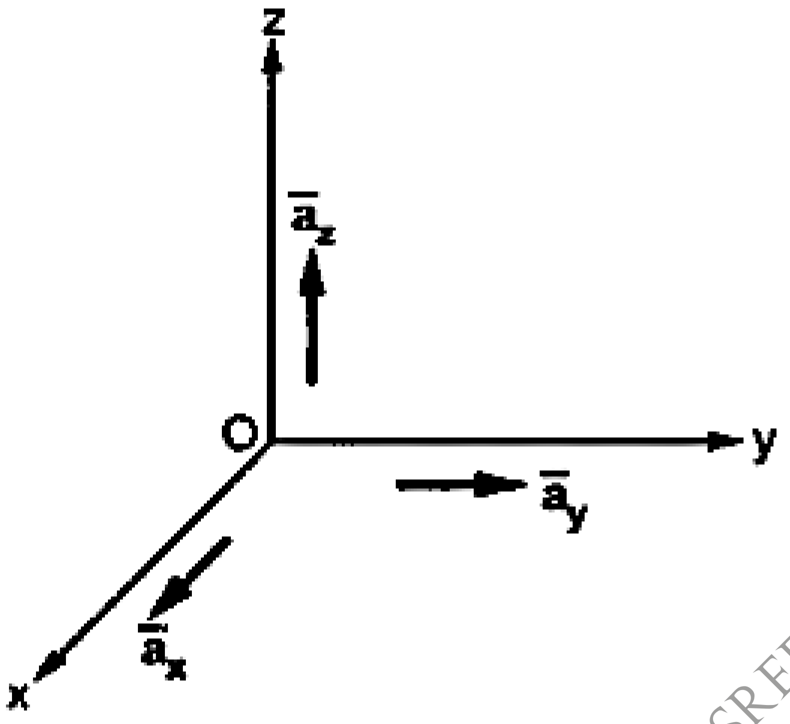



**Fig. 1.12 Unit vectors in cartesian system**

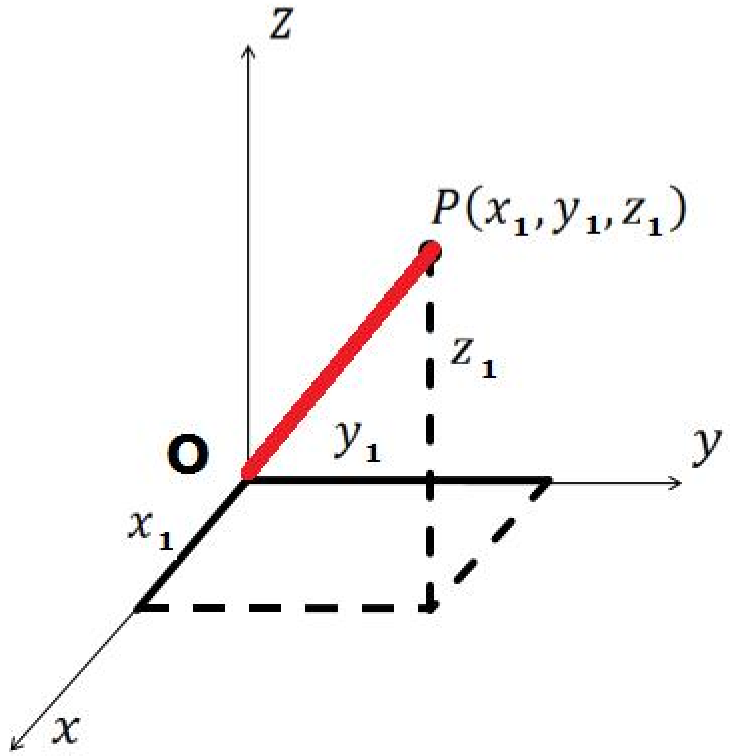
Thus for cartesian co-ordinate system, the three base vectors are the unit vectors oriented in x, y and z axis of the system. So $\bar{a}_x$, $\bar{a}_y$ and $\bar{a}_z$ are the base vectors of cartesian co-ordinate system. These are shown in the Fig. 1.12.

So any point on x-axis having co-ordinates $(x_1, 0, 0)$ can be represented by a vector joining origin to this point and denoted as $x_1 \bar{a}_x$.

The base vectors are very important in representing a vector in terms of its components, along the three co-ordinate axes.

### 1.6.3 Position and Distance Vectors

**Position Vector**




$$\boxed{\bar{r}_{OP} = x_1\,\bar{a}_x + y_1\,\bar{a}_y + z_1\,\bar{a}_z}$$

$$\boxed{|\bar{r}_{OP}| = \sqrt{(x_1)^2 + (y_1)^2 + (z_1)^2}}$$

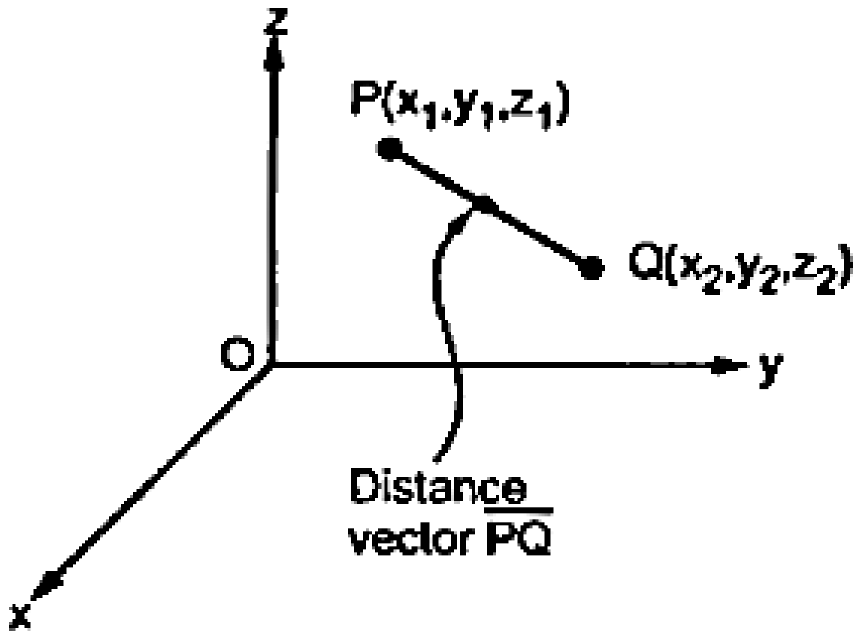


Thus if point P has co-ordinates (1, 2, 3) then its position vector is,

$$\bar{r}_{OP} = 1\ \bar{a}_x + 2\ \bar{a}_y + 3\ \bar{a}_z$$

and $|\bar{r}_{OP}| = \sqrt{(1)^2 + (2)^2 + (3)^2} = \sqrt{14} = 3.7416$

## Distance vector




$$\bar{P} = x_1 \bar{a}_x + y_1 \bar{a}_y + z_1 \bar{a}_z$$

$$\bar{Q} = x_2 \bar{a}_x + y_2 \bar{a}_y + z_2 \bar{a}_z$$

$$\boxed{\overline{PQ} = \bar{Q} - \bar{P} = [x_2 - x_1]\,\bar{a}_x + [y_2 - y_1]\,\bar{a}_y + [z_2 - z_1]\,\bar{a}_z}$$

$$\boxed{|\overline{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

$$\boxed{\bar{a}_{PQ} = \text{Unit vector along PQ} = \frac{\overline{PQ}}{|\overline{PQ}|}}$$

Let us summarize procedure to obtain distance vector and unit vector.

**Step 1 :** Identify the direction of distance vector i.e. starting point $(x_1, y_1, z_1)$ and terminating point $(x_2, y_2, z_2)$.

**Step 2 :** Subtract the respective co-ordinates of starting point from terminating point. These are three components of distance vector i.e. $(x_2 - x_1)\,\bar{a}_x$, $(y_2 - y_1)\,\bar{a}_y$ and $(z_2 - z_1)\,\bar{a}_z$

**Step 3 :** Adding the three components distance vector can be obtained.

**Step 4 :** Calculate the magnitude of the distance vector using equation (4).

**Step 5 :** Unit vector along the distance vector can be obtained by using equation (5).

*Obtain the unit vector in the direction from the origin towards the point* $P(3,-3,-2)$.

**Solution :** The origin O (0, 0, 0) while P (3, – 3, – 2) hence the distance vector $\overline{OP}$ is,

$$\overline{OP} = (3-0)\bar{a}_x + (-3-0)\bar{a}_y + (-2-0)\bar{a}_z = 3\bar{a}_x - 3\bar{a}_y - 2\bar{a}_z$$

$$\therefore \qquad |\overline{OP}| = \sqrt{(3)^2 + (-3)^2 + (-2)^2} = 4.6904$$

Hence the unit vector along the direction OP is,

$$\bar{a}_{OP} = \frac{\overline{OP}}{|\overline{OP}|} = \frac{3\bar{a}_x - 3\bar{a}_y - 2\bar{a}_z}{4.6904}$$

$$= 0.6396\,\bar{a}_x - 0.6396\,\bar{a}_y - 0.4264\,\bar{a}_z$$

*Two points $A(2,2,1)$ and $B(3,-4,2)$ are given in the cartesian system. Obtain the vector from A to B and a unit vector directed from A to B.*

**Solution :** The starting point is A and terminating point is B.

Now $\quad \bar{A} = 2\,\bar{a}_x + 2\bar{a}_y + \bar{a}_z$ and $\bar{B} = 3\bar{a}_x - 4\bar{a}_y + 2\bar{a}_z$

$$\therefore \quad \overline{AB} = \bar{B} - \bar{A} = (3-2)\bar{a}_x + (-4-2)\bar{a}_y + (2-1)\bar{a}_z$$

$$\therefore \quad \overline{AB} = \bar{a}_x - 6\,\bar{a}_y + \bar{a}_z$$

This is the vector directed from A to B.

Now $\left|\overline{AB}\right| = \sqrt{(1)^2 + (-6)^2 + (1)^2} = 6.1644$

Thus unit vector directed from A to B is,

$$\bar{a}_{AB} = \frac{\overline{AB}}{\left|\overline{AB}\right|} = \frac{\bar{a}_x - 6\bar{a}_y + \bar{a}_z}{6.1644}$$

$$= 0.1622\,\bar{a}_x - 0.9733\,\bar{a}_y + 0.1622\,\bar{a}_z$$

# UNIT - I
# ELECTROSTATICS

MB'S LAW:- UNIT-I 

CHARGE:- A point charge means that electric charge which readed on a surface & space whose geometrical dimensions very very small compared to the other dimension, in which effect of its electric field is to be studied.

Thus a point charge has a location but not the dimension. charge can be +ve & -ve. An electron possesses a -ve charge, the deficiency of an electron produces +ve charge while excess an electron produces -ve charge.

It is measured in coulombs (C).

TEMENT OF COULOMB'S LAW:

The coulomb's law states that force b/w the two point rges $Q_1$ & $Q_2$,

acts along the line joining the two point charges

is directly proportional to the product of the two charges

is inversely proportional to the square of the distance b/w them.

$Q_1 \longleftrightarrow Q_2$, $R$

Consider two point charges $Q_1$ & $Q_2$ as shown in fig. separated by the distance R. The charge $Q_1$ exerts a force on $Q_2$ while $Q_2$ also exerts a force on $Q_1$. The force acts along the line joining $Q_1$ and $Q_2$.

The force exerted b/w them is repulsive if the charges are of same polarity.

$$F \propto \frac{Q_1 Q_2}{R^2}$$

$$F = K \frac{Q_1 Q_2}{R^2}$$

$K$ = Constant of proportionality - effect of medium in which charges are located.

$$K = \frac{1}{4\pi\varepsilon}$$

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

$\varepsilon_0$ = permittivity of free space

$\varepsilon_r$ = Relative permittivity of the medium wrt free sp

$\varepsilon$ = Absolute permittivity.

for the free space or vacuum, $\varepsilon_r = 1$

$$\varepsilon = \varepsilon_0$$

$$F = \frac{1}{4\pi\varepsilon_0} \frac{Q_1 Q_2}{R^2}$$

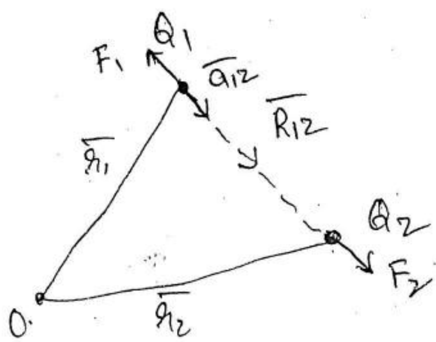
$$\varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m}$$

$$\boxed{F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2} \text{ N}}$$



## VECTOR FORM OF COULOMB'S LAW:-

The force exerted b/w the two point charges has a fixed direction which is a straight line joining the two charges. Hence the force exerted b/w the two charges can be expressed in a vector form.




Consider the two point charges $Q_1$ & $Q_2$ located at the points having position vectors $\bar{r}_1$ and $\bar{r}_2$ as shown in fig.

Then the force exerted by $Q_1$ & $Q_2$ acts along the direction ②

$\bar{R}_{12}$ where $\bar{a}_{12}$ is unit vector along $\bar{R}_{12}$.

$$\bar{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\bar{a}_{12}$$

$$\bar{a}_{12} = \text{unit vector along } \bar{R}_{12}$$

$$= \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{\bar{r}_2 - \bar{r}_1}{|\bar{R}_{12}|} = \frac{\bar{r}_2 - \bar{r}_1}{|\bar{r}_2 - \bar{r}_1|}$$

NOTE:-

1) $F_1$ is the force exerted on $Q_1$ due to $Q_2$

$$\bar{F}_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{21}^2}\bar{a}_{21}$$

$$= \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{21}^2} = \frac{\bar{r}_1 - \bar{r}_2}{|\bar{r}_1 - \bar{r}_2|}$$

$$\therefore \bar{r}_1 - \bar{r}_2 = -[\bar{r}_2 - \bar{r}_1]$$

$$\bar{a}_{21} = -\bar{a}_{12}$$

$$\therefore \bar{F}_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{21}^2}(-\bar{a}_{12}) = -\bar{F}_2$$

Hence force exerted by the two charges on each other is equal but opposite in direction.

2) Like charges repel each other, unlike charges attract each other.

3) It is necessary that two charges are the point charges and stationary in nature.

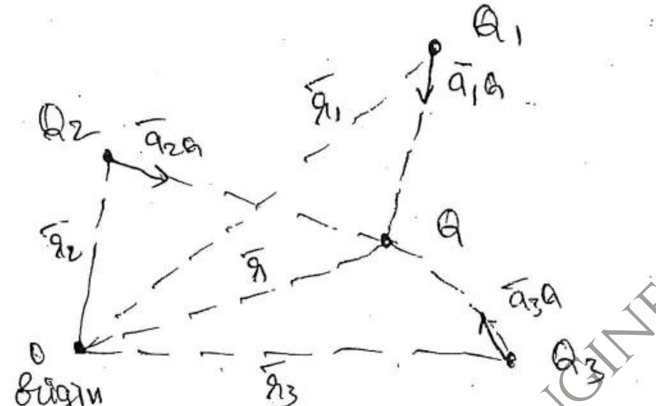
4) Two point charges may be positive or negative. Hence their signs must be considered.

i) Coulomb's law is linear which shows that if any one charge is increased 'n' times then the force exerted also increases by n times. $\bar{F}_2 = -\bar{F}_1$ then $n\bar{F}_2 = -n\bar{F}_1$ $n$ = SCALAR.

PRINCIPLE OF SUPERPOSITION:-

If there are more than two point charges, then each will exert force on the other, then the net force on any charge can be obtained by the principle of superposition.

Consider a point charge $Q$ surrounded by three other point charges $Q_1$, $Q_2$ and $Q_3$ as shown in fig.




The total force on $Q$ in such a case is vector sum of all the forces exerted on $Q$ due to each of the other point charges, $Q_1$, $Q_2$ & $Q_3$.

Consider force exerted on $Q$ due to $Q_1$. according to principle of superposition effects of $Q_2$ & $Q_3$ are to be suppressed.

$$\therefore \bar{F}_{Q_1Q} = \frac{Q_1 Q}{4\pi\varepsilon_0 \bar{R}_{1Q}} \bar{a}_{1Q} \Rightarrow \bar{a}_{1Q} = \frac{\bar{r} - \bar{r}_1}{|\bar{r} - \bar{r}_1|}$$

$$\Rightarrow \bar{F}_{Q_2Q} = \frac{Q_2 Q}{4\pi\varepsilon_0 \bar{R}_{2Q}} \bar{a}_{2Q} \Rightarrow \bar{a}_{2Q} = \frac{\bar{r} - \bar{r}_2}{|\bar{r} - \bar{r}_2|}$$

$$\Rightarrow \bar{F}_{Q_3Q} = \frac{Q_3 Q}{4\pi\varepsilon_0 \bar{R}_{3Q}} \bar{a}_{3Q} \Rightarrow \bar{a}_{3Q} = \frac{\bar{r} - \bar{r}_3}{|\bar{r} - \bar{r}_3|}$$

$$\bar{F}_Q = \bar{F}_{Q_1Q} + \bar{F}_{Q_2Q} + \bar{F}_{Q_3Q}$$

in general if there are 'n' other charges then force exerted on $Q$ due to all other 'n' charges is, $\bar{F}_Q = \bar{F}_{Q_1Q} + \bar{F}_{Q_2Q} + \cdots + \bar{F}_{Q_nQ}$.

$$\boxed{\bar{F}_Q = \frac{Q}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{Q_i}{R_{iQ}^2} \frac{\bar{r} - \bar{r}_i}{|\bar{r} - \bar{r}_i|}}$$ Newton.

TEPS TO SOLVE PROBLEMS ON COULOMB'S LAW:-

TEP-1: Obtain the position vectors of the points where the charges are located.

TEP-2: Obtain the unit vector along the straight line joining the charges. The direction is towards the charge on which the force exerted is to be calculated.

TEP-3: Using coulomb's law, express the force exerted in the vector form.

TEP-4: If there are more charges, repeat steps 1 to 3 for each charge exerting a force on the charge under consideration

TEP-5: Using the principle of superposition, the vector sum of all the forces calculated earlier is the resultant force, exerted on the charge under consideration.

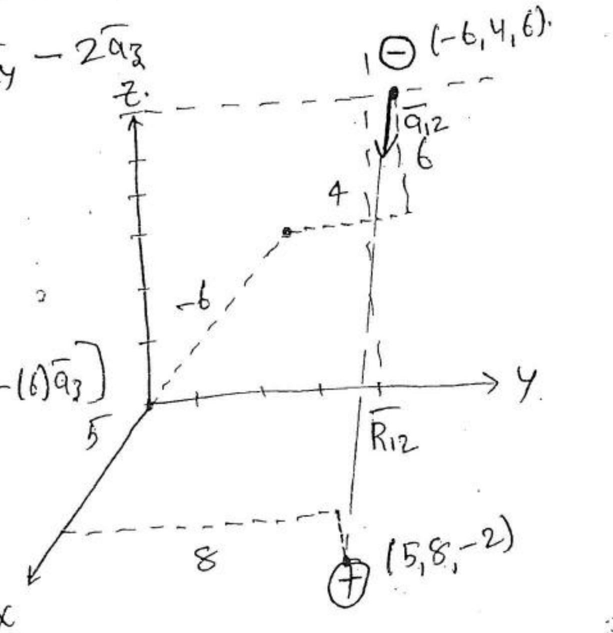
) A charge $Q_1 = -20\mu C$ is located at $P(-6,4,6)$ and a charge $Q_2 = 50\mu C$ is located at $R(5,8,-2)$ in a free space. Find the force exerted on $Q_2$ by $Q_1$ in vector form. The distances given are in metre.



!: $P(-6,4,6)$ & $R(5,8,-2)$

position vectors $\Rightarrow \bar{P} = -6\bar{a}_x + 4\bar{a}_y + 6\bar{a}_z$

$\bar{R} = 5\bar{a}_x + 8\bar{a}_y - 2\bar{a}_z$




$$\bar{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\bar{a}_{12}$$

$\bar{R}_{12} = \bar{R}_{PR} = \bar{R} - \bar{P}$

$= [5-(-6)]\bar{a}_x + (8-4)\bar{a}_y + [-2-(6)\bar{a}_z]$

$= 11\bar{a}_x + 4\bar{a}_y - 8\bar{a}_z$

$|R_{12}| = \sqrt{11^2 + 4^2 + -8^2}$

$= 14.1774$

$$\bar{a}_{12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{11\bar{a}_x + 4\bar{a}_y - 8\bar{a}_z}{14.1774}$$

$$\bar{a}_{12} = 0.7758\,\bar{a}_x + 0.2821\,\bar{a}_y - 0.5642\,\bar{a}_z$$

$$\bar{F}_2 = \frac{-20\times10^{-6}\times 50\times10^{-6}}{4\pi\times 8.854\times10^{-12}\times(14.1774)^2}\,[\bar{a}_{12}]$$

$$= -0.0447\,[0.7758\,\bar{a}_x + 0.2821\,\bar{a}_y - 0.5642\,\bar{a}_z]$$

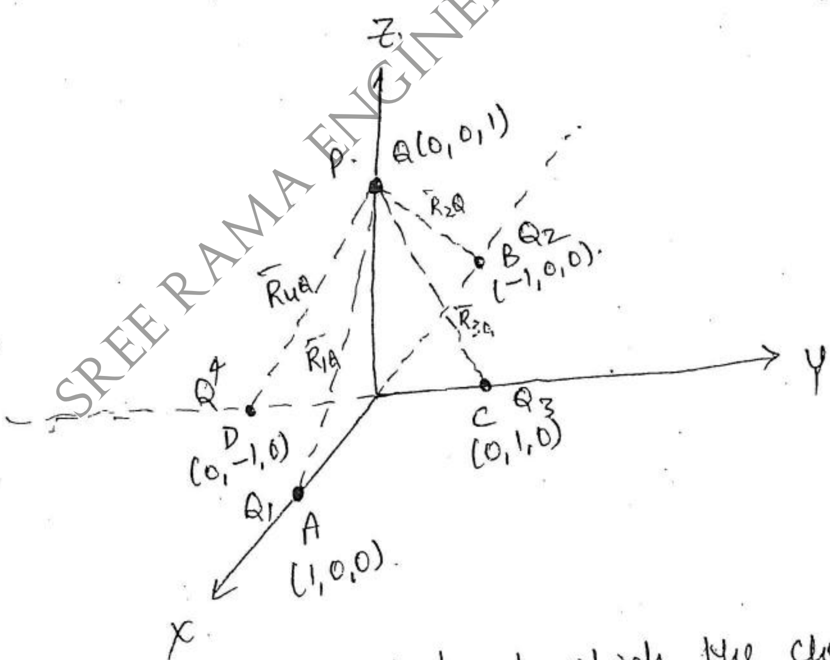
$$= -0.0346\,\bar{a}_x - 0.01261\,\bar{a}_y + 0.02522\,\bar{a}_z \quad N$$

$$\therefore |\bar{F}_2| = \sqrt{(0.0346)^2 + (0.01261)^2 + (-0.02522)^2}$$

$$= 44.634\ mN$$

Q: Four point charges each of $10\mu C$ are placed in free space at the points $(1,0,0)$, $(-1,0,0)$, $(0,1,0)$ & $(0,-1,0)$ m res. Determine the force on a point charge of $30\mu C$ located at a point $(0,0,1)$ m.

Sol:




The position vectors of 4 points at which the charges $Q_1$ to $Q_4$ are located

$\bar{A} = \bar{a}_x$ $\qquad\qquad \bar{P} = \bar{a}_z$

$\bar{B} = -\bar{a}_x$

$\bar{C} = \bar{a}_y$

$\bar{D} = -\bar{a}_y$

force on Q due to $Q_1$ alone

$$\bar{F}_1 = \frac{QQ_1}{4\pi\varepsilon_0 R_{1Q}^2} \cdot \bar{a}_{1Q} \Rightarrow \bar{a}_{1Q} = \frac{\bar{R}_{1Q}}{|\bar{R}_{1Q}|}$$

$$\bar{R}_{1Q} = \bar{P} - \bar{A} = \bar{a}_z - \bar{a}_x$$

$$|\bar{R}_{1Q}| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\therefore \bar{F}_1 = \frac{30\times10^{-6}\times10\times10^{6}}{4\pi\times8.854\times10^{-12}\times(\sqrt{2})^2}\left[\frac{\bar{a}_z - \bar{a}_x}{\sqrt{2}}\right]$$

$$= 0.9533\left[\bar{a}_z - \bar{a}_x\right] \rightarrow (1)$$

$$\bar{R}_{2Q} = \bar{P} - \bar{B} = \bar{a}_z + \bar{a}_x \Rightarrow \bar{a}_{2Q} = \frac{\bar{a}_z + \bar{a}_x}{\sqrt{2}}$$

$$\bar{R}_{3Q} = \bar{P} - \bar{C} = \bar{a}_z - \bar{a}_y \Rightarrow \bar{a}_{3Q} = \frac{\bar{a}_z - \bar{a}_y}{\sqrt{2}}$$

$$\bar{R}_{4Q} = \bar{P} - \bar{D} = \bar{a}_z + \bar{a}_y \Rightarrow \bar{a}_{4Q} = \frac{\bar{a}_z + \bar{a}_y}{\sqrt{2}}$$

$$\therefore \frac{QQ_2}{4\pi\varepsilon_0 R_{2Q}^2} = \frac{QQ_3}{4\pi\varepsilon_0 R_{3Q}^2} = \frac{QQ_4}{4\pi\varepsilon_0 R_{4Q}^2} = \frac{30\times10^{-6}\times10\times10^{-6}}{4\pi\times8.854\times10^{-12}\times(\sqrt{2})^2} = 1.3481$$

$$\bar{F}_2 = 1.3481\left[\frac{\bar{a}_z + \bar{a}_x}{\sqrt{2}}\right] = 0.9533(\bar{a}_z + \bar{a}_x) \rightarrow (2)$$

$$\bar{F}_3 = \qquad = 0.9533(\bar{a}_z - \bar{a}_y) \rightarrow (3)$$

$$\bar{F}_4 = \qquad = 0.9533(\bar{a}_z + \bar{a}_y) \rightarrow (4)$$

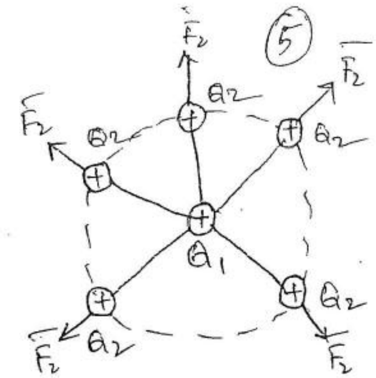
$$\therefore \bar{F}_t = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4$$

$$= 0.9533\left[\bar{a}_z - \cancel{\bar{a}_x} + \bar{a}_z + \cancel{\bar{a}_x} + \bar{a}_z - \cancel{\bar{a}_y} + \bar{a}_z + \cancel{\bar{a}_y}\right]$$

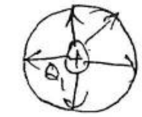
$$= 0.9533 \times 3\,\bar{a}_z$$

$$\bar{F}_t = 3.813\,\bar{a}_z \text{ Newton.}$$

## ELECTRIC FIELD INTENSITY :-




Consider a point charge $Q_1$ as shown in fig & any other similar charge $Q_2$ is brought near it, $Q_2$ experiences a force. If $Q_2$ is moved around $Q_1$, still $Q_2$ experiences a force as shown fig.




Thus there exists a region around a charge in which it exerts a force on any other charge. This region where a particular charge exerts a force on any other charge located in that region is called electric field of that charge.

The force experienced by the charge $Q_2$ due to $Q_1$ is given by Coulomb's law

$$\bar{F_2} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\bar{a}_{12}$$

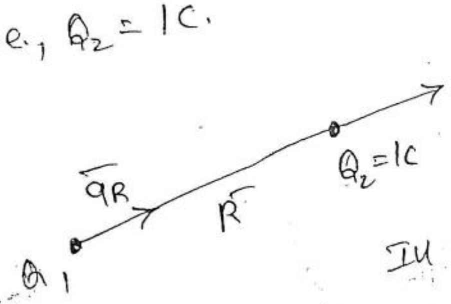
Hence force per unit charge can be written as

$$\frac{\bar{F_2}}{Q_2} = \frac{Q_1}{4\pi\varepsilon_0 R_{12}^2}\bar{a}_{12}$$

This force exerted per unit charge is called Electric field intensity or electric field strength. It is a vector quantity

$$\bar{E} = \frac{Q_1}{4\pi\varepsilon_0 R_{1P}^2}\bar{a}_{1P} \Rightarrow P = \text{position of any other charge around } Q_1$$

Another definition of EFI is the force experienced by a unit positive test charge i.e., $Q_2 = 1C$.




$$\bar{E} = \frac{Q_1}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

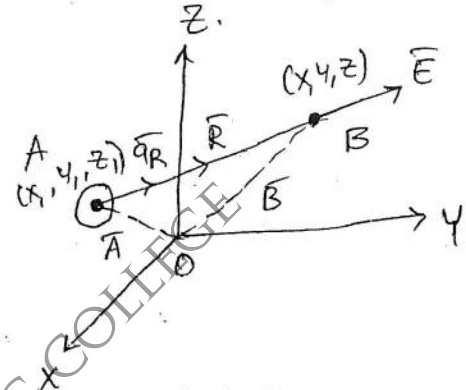
In spherical coordinate system then unit vector $\bar{a}_R$, & the distance R is the radius of the sphere r.

$$\bar{E} = \frac{Q_1}{4\pi\varepsilon_0 r^2}\bar{a}_r$$

UNITS OF $\bar{E}$: $\bar{E} = \frac{\text{Force}}{\text{Unit charge}} = \frac{N}{C}$ or $V/m$

METHOD OF OBTAINING $\bar{E}$ IN CARTESIAN SYSTEM:-

Consider a charge Q located at point $A(x_1, y_1, z_1)$ as shown in fig. It is required to obtain $\bar{E}$ at any point $B(x, y, z)$ in the cartesian system. Then $\bar{E}$ at point B can be obtained using following steps.




STEP-1:-
obtain the position vectors of points A & B

$\therefore \bar{r}_A = \bar{A}$ while

$\bar{r}_B = \bar{B}$

$\therefore \bar{A} = x_1\bar{a}_x + y_1\bar{a}_y + z_1\bar{a}_z$

$\bar{B} = x\bar{a}_x + y\bar{a}_y + z\bar{a}_z$

STEP-2:- Find the distance vector $\bar{R}$ directed from A to B.

$$\bar{R} = \bar{B} - \bar{A} = (x - x_1)\bar{a}_x + (y - y_1)\bar{a}_y + (z - z_1)\bar{a}_z$$

STEP-3:- Find the unit vector $\bar{a}_R$ along the direction from A to B

$$\bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{\bar{B}-\bar{A}}{|\bar{B}-\bar{A}|}$$

STEP-4:- obtain the $\bar{E}$ at the point B as,

$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\bar{a}_R = \frac{Q}{4\pi\varepsilon_0 R^2}\frac{\bar{R}}{|\bar{R}|} \quad V/m$$

STEP-5:- Magnitude of $\bar{E}$ is given by,

$$|\bar{E}| = \frac{Q}{4\pi\varepsilon_0 R^2} \quad V/m.$$

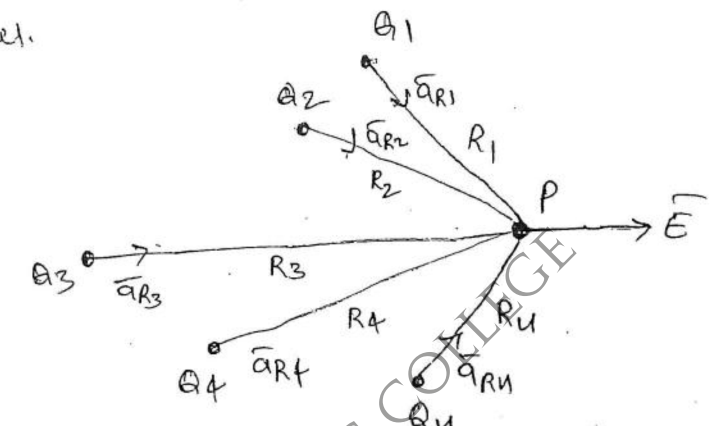
Substituting $\bar{R}$ and $|\bar{R}|$ interms of the Cartesian coordinates of A & B, the required EFI $\bar{E}$ at the point B can be obtained.

## LECTRIC FIELD DUE TO DISCRETE CHARGES:-



The electric field at a point due to 'n' number of charges is to be obtained using law of superposition.

Consider 'n' charges $Q_1, Q_2, \ldots Q_n$ as shown in fig. The combined EFI is to be obtained at point P. The distances of point P from $Q_1, Q_2, \ldots Q_n$ are $R_1, R_2, \ldots R_n$ resp. The unit vectors along these directions are $\bar{a}_{R1}, \bar{a}_{R2}, \ldots \bar{a}_{Rn}$ resp.




$$\bar{E} = \bar{E}_1 + \bar{E}_2 + \ldots \bar{E}_n$$

$$= \frac{Q_1}{4\pi\varepsilon_0 R_1^2}\bar{a}_{R1} + \frac{Q_2}{4\pi\varepsilon_0 R_2^2}\bar{a}_{R2} + \ldots + \frac{Q_n}{4\pi\varepsilon_0 R_n^2}\bar{a}_{Rn}$$

$$\bar{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{Q_i}{R_i^2}\bar{a}_{Ri}$$

$$\bar{a}_{Ri} = \frac{\bar{r}_p - \bar{r}_i}{|\bar{r}_p - \bar{r}_i|}$$

$\bar{r}_p$ – position vector of point P

$\bar{r}_i$ – position vector of point where charge $Q_i$ is placed.

NOTE:-

1) $\bar{E}$ around a charge $Q_1$ is directly proportional to the charge $Q_1$
2) $\bar{E}$ " " inversely " " distance b/w charge $Q_1$ and point at which $\bar{E}$ is to be calculated.
3) $\bar{E}$ at any point & $\bar{F}$ exerted on a charge placed at the same point are always in the same direction.
4) placing unit charge is a method of detecting the presence of electric field around a charge. without any unit test charge placed nearby, every charge has its electric field always existing around it.
5) The test charge placed must be small enough so that the EFI to be ... should not get disturbed.

## TYPES OF CHARGE DISTRIBUTIONS:-

There are four types of Charge distributions

(1) point charge

(2) Line charge

(3) surface charge

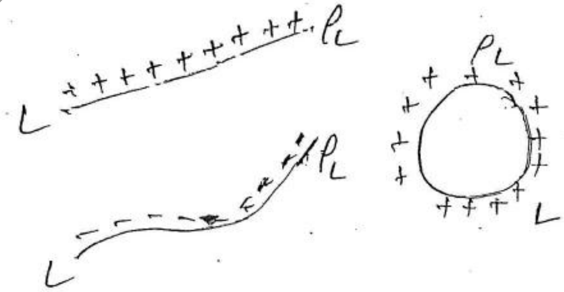
(4) volume charge.

### 1 POINT CHARGE :-

If the dimensions of a surface carrying charge are very very small compared to region surrounding it then the surface can be treated to be a point. The corresponding charge is called point charge.

### 1 LINE CHARGE:-

It is possible that the charge may be spreaded all along a line, which be finite or infinite. Such a charge uniformly distributed along a line is called a line charge.




The charge density of the line charge is denoted as $\rho_L$ and defined as charge per unit length

$$\rho_L = \frac{\text{Total charge in Coulomb}}{\text{Total length in metres}} \ C/m$$

$\rho_L$ is const all along the length L of the line carrying the charge

In many cases, $\rho_L$ is given to be the function of coordinates of the line i.e, $\rho_L = 3x$ or $\rho_L = 4y^2$ etc., In such a case it is necessary to find the total charge Q by considering differential length dl of the line. Then by integrating the charge dQ on dl, for the entire length, total charge Q is to be obtained, such an integral is called 'line integral'

$$dQ = \rho_L \, dl$$

sharp beam in a CRT

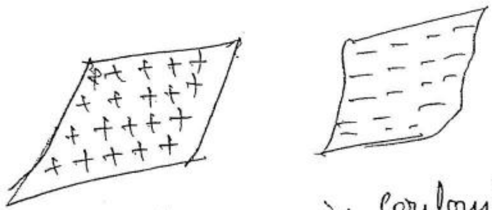
$$Q = \int_L dQ \Rightarrow \boxed{Q = \int_L \rho_L \, dl}$$

If the line of length L is a closed path, then integral is called closed contour integral

$$\boxed{Q = \oint \rho_L \, dl}$$

**RFACE CHARGE:-**

If the charge is distributed uniformly over a two dimensional surface then it is called surface charge or a sheet of charge.



$dQ = \rho_S \, ds$

$ds$ - surface area

$$\rho_S = \frac{\text{Total charge in coulomb}}{\text{Total area in sq. m}} \quad (C/m^2)$$

$\rho_S$ is const. over the surface carrying the charge.

$$Q = \int_S dQ = \int_S \rho_S \, ds$$

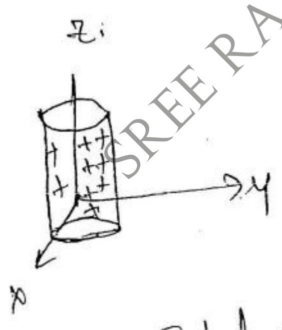
$$\boxed{Q = \int_S \rho_S \, ds}$$

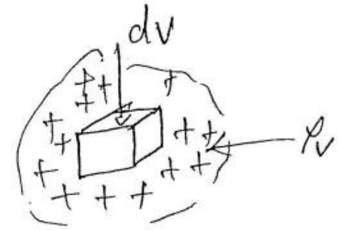
plate of a charged parallel plate capacitor

if dimensions of the sheet of charge are very large compared to the distance then it is called infinite sheet of charge.

**LUME CHARGE:-**

If the charge is distributed uniformly in a volume then it is called volume charge.




$dQ = \rho_V \, dv$

$dv$ - differential volume.

$$\rho_V = \frac{\text{Total charge in coulomb}}{\text{Total volume in cubic metre}} \quad (C/m^3)$$

$$\boxed{Q = \int_{vol} \rho_V \, dV}$$

charged cloud is an example of volume charge.

OTE: → $\rho_S$ & $\rho_V$ can be functions of coordinate system.

→ surface integral double integration

triple integration

Q:- Find the total charge inside a volume having volume charge density as $10z^2 e^{-0.1x} \sin \pi y$ $C/m^3$. The volume is defined b/w $-2 \le x \le 2$, $0 < y \le 1$ & $3 \le z \le 4$

A:- $$\rho_V = 10z^2 e^{-0.1x} \sin \pi y$$

$$dv = dx\,dy\,dz$$

$$dQ = \rho_V\, dV$$

$$Q = \int_{vol} \rho_V\, dV = \int_{z=3}^{4}\int_{0}^{1}\int_{-2}^{2} 10z^2 e^{-0.1x} \sin \pi y \; dx\,dy\,dz$$

$$= \int_{z=3}^{4}\int_{y=0}^{1} 10z^2 \sin \pi y \left[\frac{e^{-0.1x}}{-0.1}\right]_{-2}^{2} dy\,dz$$

$$= \int_{z=3}^{4} 10z^2 \left[-\frac{\cos \pi y}{\pi}\right]_0^1 \left[\frac{e^{-0.2}}{-0.1} - \frac{e^{+0.2}}{-0.1}\right] dz.$$

$$= 10\left[\frac{z^3}{3}\right]_3^4 \left[\frac{-\cos \pi}{\pi} - \frac{-\cos 0}{\pi}\right] 4.0267$$

$$= 10\left(\frac{4^3 - 3^3}{3}\right)\left[\frac{1}{\pi} + \frac{1}{\pi}\right] 4.0267$$

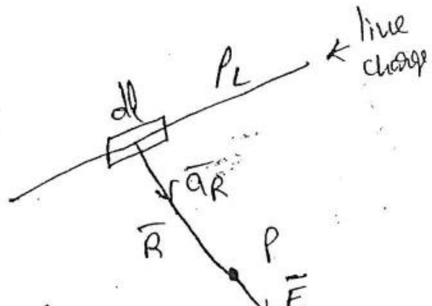
$$= 316.162\ C.$$



## EFI DUE TO VARIOUS CHARGE DISTRIBUTIONS:-

$\bar{E}$ due to point charge: $\bar{E} = \dfrac{Q}{4\pi\varepsilon_0 R^2}\, \bar{a}_R$

$\bar{E}$ due to Line Charge:




Consider a line charge distribution having a charge density $\rho_L$

The charge dQ on the differential length dl

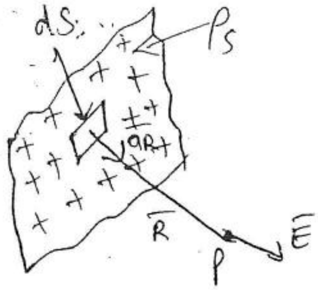
$$dQ = \rho_L\, dl$$

$$\therefore d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\, \bar{a}_R = \frac{\rho_L\, dl}{4\pi\varepsilon_0 R^2}\, \bar{a}_R$$

$\bar{a}_R$ & dl is to be obtained depending upon the coordinate

$$\bar{E} = \int \frac{\rho_L\, dl}{\ldots}\, \bar{a}_R$$

DUE TO SURFACE CHARGE:-




$\rho_S$.

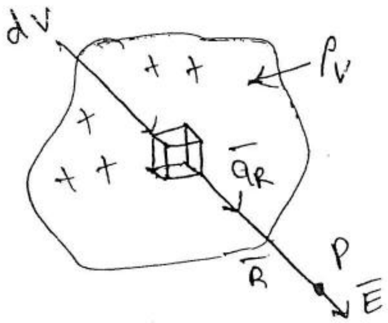
$dQ = \rho_S\, ds$.

$$d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\bar{a}_R = \frac{\rho_S\, ds}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

$$\bar{E} = \int_S \frac{\rho_S\, ds}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

$\bar{a}_R$ & ds to be obtained according to the position of the sheet of charge & the coordinate system used.

DUE TO VOLUME CHARGE:-




$\rho_V$

$dQ = \rho_V\, dV$

$$d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\bar{a}_R = \frac{\rho_V\, dV}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

$$\bar{E} = \int_{Vol} \frac{\rho_V\, dV}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

$\bar{a}_R$ and dv must be obtained according to the Co-ordinate system used

If there are all possible types of charge distribution,

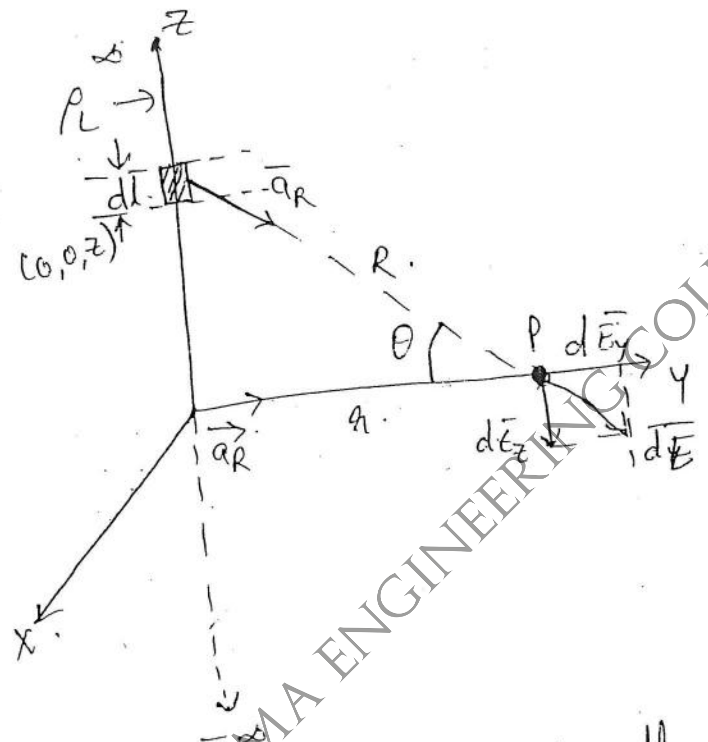
then the total $\bar{E}$ at a point is

$$\bar{E}_{total} = \bar{E}_P + \bar{E}_l + \bar{E}_S + \bar{E}_V$$

## ECTRIC FIELD DUE TO INFINITE LINE CHARGE:-

Consider an infinitely long straight line carrying uniform line arge having density $\rho_L$ c/m. Let this line lies along z-axis from $-\infty$ to $\infty$ and hence called infinite line charge.

Let point P is on y axis at which electric field intensity is to be etermined. The distance from the origin is r.




Consider a small differential length dl carrying a charge dQ, along the line as shown in fig. It is along z axis hence $dl = dz$

$$dQ = \rho_L \, dl$$
$$= \rho_L \, dz$$

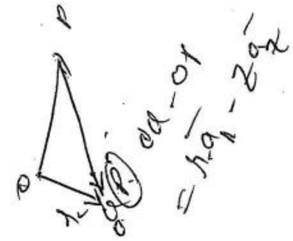
The coordinates of dQ are $(0, 0, z)$ while the coordinates of P are $(0, r, 0)$.

hence the distance vector $\bar{R} = \bar{r}_p - \bar{r}_{dl}$

$$= \left[ r\bar{a}_y - z\bar{a}_z \right]$$

$$|\bar{R}| = \sqrt{r^2 + z^2}$$

$$\therefore \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{r\bar{a}_y - z\bar{a}_z}{\sqrt{r^2+z^2}}$$



$$\therefore d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\,\bar{a}_R$$

$$= \frac{\rho_L\, dz}{4\pi\varepsilon_0\left(\sqrt{r^2+z^2}\right)^2}\cdot\left[\frac{r\bar{a}_y - z\bar{a}_z}{\sqrt{r^2+z^2}}\right]$$

$\therefore$ NOTE: For every charge on +ve z-axis there is equal charge present on −ve z-axis. Hence the z component of EFI produced by such charges at point P will cancel each other.

$$d\bar{E} = \frac{\rho_L\, dz}{4\pi\varepsilon_0\left(\sqrt{r^2+z^2}\right)^2}\cdot\frac{r\bar{a}_y}{\sqrt{r^2+z^2}}$$

now $\bar{E}$ at point P.

$$\bar{E} = \int_{-\infty}^{\infty} \frac{\rho_L}{4\pi\varepsilon_0\left(r^2+z^2\right)^{3/2}}\cdot r\,dz\,\bar{a}_y$$

$$z = r\tan\theta \quad (\text{or}) \quad r = \frac{z}{\tan\theta}.$$

$$dz = r\sec^2\theta\, d\theta.$$

for $z = -\infty.\quad \theta = \tan^{-1}(-\infty) = -\pi/2 = -90^\circ.$

$z = +\infty \quad \theta = \tan^{-1}(\infty) = \pi/2 = 90^\circ.$

$$\therefore \bar{E} = \int_{\theta=-\pi/2}^{\pi/2} \frac{\rho_L}{4\pi\varepsilon_0\left[r^2+r^2\tan^2\theta\right]^{3/2}}\cdot r\cdot r\sec^2\theta\cdot d\theta\,\bar{a}_y$$

$$= \frac{\rho_L}{4\pi\varepsilon_0}\int_{-\pi/2}^{\pi/2}\frac{r^2\sec^2\theta\, d\theta}{r^3\left(1+\tan^2\theta\right)^{3/2}}\,\bar{a}_y$$

$1+\tan^2\theta = \sec^2\theta$

$$\bar{E} = \frac{\rho_L}{4\pi\varepsilon_0}\int_{-\pi/2}^{\pi/2} \frac{\sec^2\theta\, d\theta}{r\sec^3\theta}\,\bar{a_y}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r}\int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta\, \bar{a_y}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r}\left[\sin\theta\right]_{-\pi/2}^{\pi/2} \cdot \bar{a_y}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r}\left[\sin\pi/2 - \sin(-\pi/2)\right]\bar{a_y}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r}\left[1-(-1)\right]\bar{a_y}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r} \times 2\bar{a_y}$$

$$\boxed{\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\,\bar{a_y}\ \text{V/m}}$$



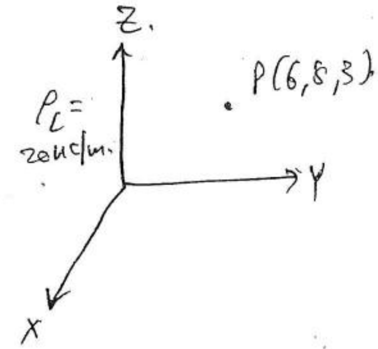
(ii) $$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\,\bar{a_r}$$

$r$ = $\perp^r$ distance of point P from the line charge

$\bar{a_r}$ = Unit vector in the direction of the $\perp^r$ distance of point P from the line charge.

NOTE: (1) EFI, $\bar{E}$ at any point has no component in the direction parallel to the line along which the charge is located and the charge is infinite.

Q: A uniform line charge, infinite in extent with $\rho_L = 20nC/m$ lies along the Z axis. Find the $\bar{E}$ at (6, 8, 3)m.




Sol:

$$\bar{r} = (6-0)\bar{a}_x + (8-0)\bar{a}_y + 0$$

$$\bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{6\bar{a}_x + 8\bar{a}_y}{\sqrt{6^2+8^2}}$$

$$= 0.6\bar{a}_x + 0.8\bar{a}_y$$

$$\bar{E} = \frac{\rho_L}{2\pi\epsilon_0 r}\bar{a}_r$$

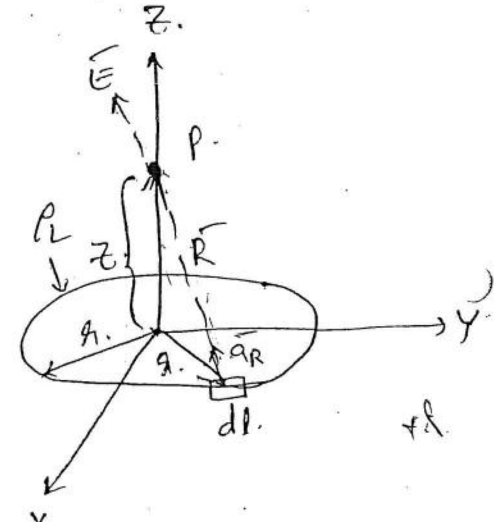
$$= \frac{20\times10^{-9}}{2\pi\times8.854\times10^{-12}\times10}\left[0.6\bar{a}_x + 0.8\bar{a}_y\right]$$

$$= 10.7853\bar{a}_x + 14.38\bar{a}_y \quad V/m.$$

ELECTRIC FIELD DUE TO CHARGED CIRCULAR RING:-

Consider a charged circular ring of radius r placed in xy plane with centre at origin, carrying a charge uniformly along its circumference. The charge density is C/m.

The point P is at a $\perp^r$ distance 'z' from the ring as shown in fig.




Consider a small differential length dl on this ring. The charge on it is dQ.

$$\therefore dQ = \rho_L\, dl$$

$$\therefore d\bar{E} = \frac{\rho_L\, dl}{4\pi\epsilon_0 R^2}\bar{a}_R$$

R = distance of point P from dl

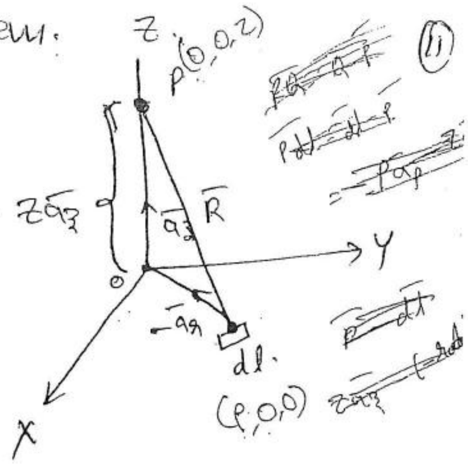
(11)

Consider the cylindrical coordinate system.

for $dl$ we are moving in $\phi$ direction

where $dl = r\,d\phi$

$$R^2 = r^2 + z^2$$




$\bar{R}$ can be obtained from its 2 components, in cylindrical system

(1) distance $r$ in the direction of $\underline{\underline{-\bar{a}_r}}$ radially inwards i.e., $-r\bar{a}_r$

(2) " $z$ " $\bar{a}_z$ i.e. $z\bar{a}_z$ $\quad -\rho\vec{a}_\rho + z\vec{a}$

$$\bar{R} = -r\bar{a}_r + z\bar{a}_z$$

$$|\bar{R}| = \sqrt{(-r)^2 + z^2} = \sqrt{r^2+z^2}$$

$$\bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2+z^2}}$$

x y z
ρ φ z
r φ θ

$\bar{R} = \bar{r}_P - \bar{r}_{dl} = P_\rho\bar{a}_\rho + P_\phi\bar{a}_\phi + P_z\bar{a}_z$

$= -\rho\bar{a}_r + 0 + 0$

coordinates of P.

$\bar{r}_P = z\bar{a}_z$

$\bar{r}_{dl} = -\rho\bar{a}_\rho$

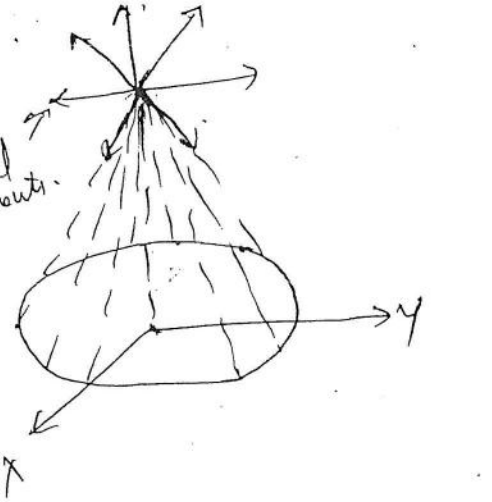
$\bar{R} = -\rho\bar{a}_\rho - z\bar{a}_z$

$$d\bar{E} = \frac{\rho_L\,dl}{4\pi\varepsilon_0\left(\sqrt{r^2+z^2}\right)^2} \times \frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2+z^2}}$$

$$d\bar{E} = \frac{\rho_L\,r\,d\phi}{4\pi\varepsilon_0(r^2+z^2)^{3/2}}\left[-r\bar{a}_r + z\bar{a}_z\right]$$

NOTE: The radial components of $\bar{E}$ at point P will be symmetrical, placed in the plane parallel to xy plane and are going to cancel each other. hence neglecting $\bar{a}_r$ component.

$$d\bar{E} = \frac{\rho_L(r\,d\phi)}{4\pi\varepsilon_0(r^2+z^2)^{3/2}}\,z\bar{a}_z$$




$$\bar{E} = \int_{\phi=0}^{2\pi} \frac{\rho_L\,r\,d\phi}{4\pi\varepsilon_0(r^2+z^2)^{3/2}}\cdot z\bar{a}_z$$

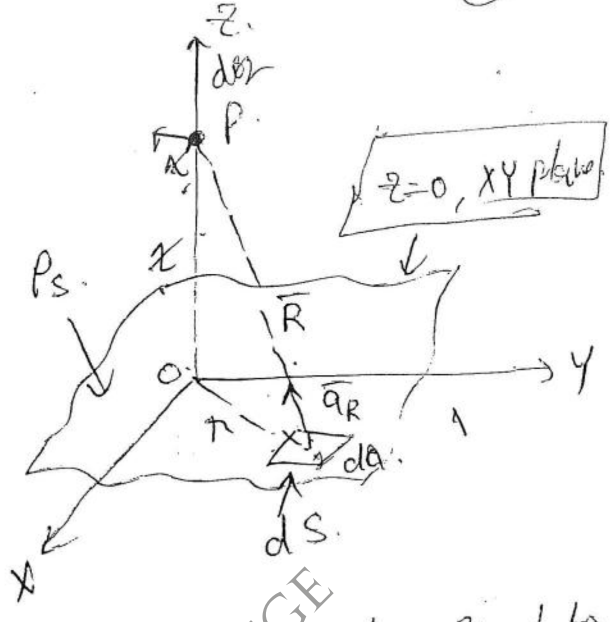
$$= \frac{\rho_L r}{4\pi\varepsilon_0(r^2+z^2)^{3/2}} z\bar{a_z} [\phi]_0^{2\pi}$$

$$\bar{E} = \frac{\rho_L r z}{2\varepsilon_0(r^2+z^2)^{3/2}} \bar{a_z}$$

$r$ = Radius of the ring

$z$ = $\perp^r$ distance of point P from the ring along the axis of the ring.

## ECTRIC FIELD DUE TO INFINITE SHEET OF CHARGE:- (12)



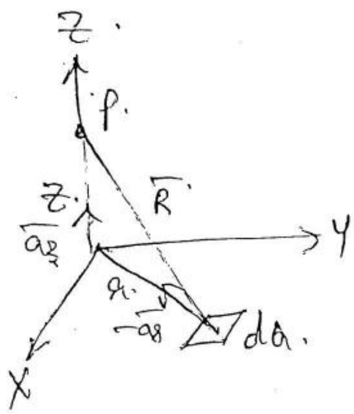

Consider an infinite sheet of charge having uniform charge density $\rho_s$ C/m$^2$, placed in XY plane.

Let us use cylindrical coordinates.

The point P at which $\bar{E}$ to be calculated is on z-axis.

Consider the differential surface area ds carrying a charge dQ. The normal direction to ds is z direction hence ds normal to z-direction is $r\,dr\,d\phi$.

$$dQ = \rho_s\, ds = \rho_s\, r\, dr\, d\phi$$



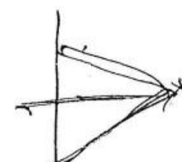

$$d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\,\bar{a}_R$$

$$= \frac{\rho_s\, r\,dr\,d\phi}{4\pi\varepsilon_0 R^2}\,\bar{a}_R$$

The distance vector $\bar{R}$ has 2 components.

(i) The radial component r along $-\bar{a}_r$ i.e., $-r\bar{a}_r$

(ii) The component z along $\bar{a}_z$ i.e., $z\bar{a}_z$



$$\bar{R} = -r\bar{a}_r + z\bar{a}_z$$

$$|\bar{R}| = \sqrt{r^2+z^2}$$

$$\bar{a}_R = \frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2+z^2}}$$

$$d\bar{E} = \frac{\rho_s\, r\,dr\,d\phi}{4\pi\varepsilon_0 (\sqrt{r^2+z^2})^2}\left[\frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2+z^2}}\right]$$

$\int \frac{\rho_s\, r\, dr\, d\phi}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}} (\cos\phi\, a_x + \sin\phi$

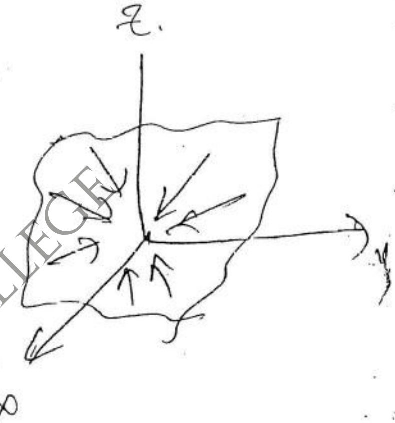
$\rho \Rightarrow (\cos\phi\, a_x + \sin\phi\, a_y)$

$(\sin\phi\, a_x\ \cos\phi\, a_y)$

$= 0$

for infinite sheet in XY plane, r varies from 0 to $\infty$ while $\phi$ varies from 0 to $2\pi$

NOTE: As there is symmetry about z-axis from all radial direction, all $\bar{a}_r$ components of $\bar{E}$ are going to cancel each other and net $\bar{E}$ will not have any radial component.
hence while integrating $d\bar{E}$ there is no need to consider $\bar{a}_r$ component. Though if considered, after integration procedure, it will get mathematically cancelled.




$$\bar{E} = \int_{\phi=0}^{2\pi}\int_{r=0}^{\infty} d\bar{E}$$

$$= \int_{\phi=0}^{2\pi}\int_{0}^{\infty} \frac{\rho_s \, r dr d\phi}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}} (z\,\bar{a}_z)$$

$r^2 + z^2 = u^2$ hence $2r dr = 2u du$

for $r = 0$, $u = z$ & $r = \infty$, $u = \infty$ changing limits.

$$E = \int_{0}^{2\pi}\int_{u=z}^{\infty} \frac{\rho_s}{4\pi\varepsilon_0} \frac{u du}{(u^2)^{3/2}} d\phi \, z\bar{a}_z$$

$$= \int_{0}^{2\pi}\int_{u=z}^{\infty} \frac{\rho_s}{4\pi\varepsilon_0} \frac{du}{u^2} d\phi \, (z\,\bar{a}_z).$$

$$= \int_{0}^{2\pi} \frac{\rho_s}{4\pi\varepsilon_0} d\phi \, z\,\bar{a}_z \left[-\frac{1}{u}\right]_z^{\infty} \quad \text{as } \int \frac{1}{u^2} = \int u^{-2} = \frac{u^{-1}}{-1} = -\frac{1}{u}$$

$$= \frac{\rho_s}{4\pi\varepsilon_0} \left[\phi\right]_0^{2\pi} (z\bar{a}_z) \left[-\frac{1}{\infty} - \left(-\frac{1}{z}\right)\right]$$

$$= \frac{\rho_s}{4\pi\varepsilon_0} (2\pi)\,\bar{a}_z$$

$$\boxed{\bar{E} = \frac{\rho_s}{2\varepsilon_0}\,\bar{a}_z} \text{ V/m}.$$

$\bar{E} = \frac{\rho_s}{2\varepsilon_0}\,\bar{a}_n$ V/m.

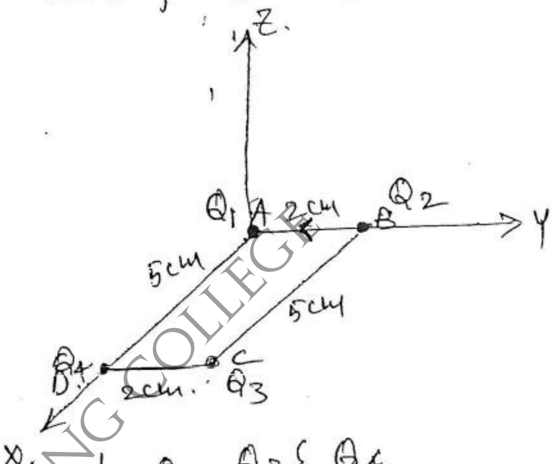
for the points below xy plane

$\bar{a}_n = -\bar{a}_n \Rightarrow \boxed{\bar{E} = -\frac{\rho_s}{2\varepsilon_0}\,\bar{a}_z}$

Four concentrated charges $Q_1 = 0.3\mu C$, $Q_2 = 0.2\mu C$, $Q_3 = 0.3\mu C$ & $Q_4 = 0.2\mu C$ are located at the vertices of a plane rectangle. The length of rectangle is 5cm & breadth of the rectangle is 2cm. Find the magnitude and direction of resultant force on $Q_1$. (13)

A charge of $-0.3\mu C$ is located at $A(25, -30, 15)$ cm and a second charge of $0.5\mu C$ is located at $B(-10, 8, 12)$ cm. Find electric field strength, E at: (1) the origin (2) at point $P(15, 20, 50)$ cm.

:(1) coordinates




$A(0, 0, 0)$.

$B(0, 0.02, 0)$

$C(0.05, 0.02, 0)$

$D(0.05, 0, 0)$

Force on $Q_1$ is sum of the forces due to $Q_2$, $Q_3$ & $Q_4$.

$$\bar{F}_{21} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 \bar{R}_{21}^2}\bar{a}_{R21} \; ; \; \bar{F}_{31} = \frac{Q_1 Q_3}{4\pi\varepsilon_0 \bar{R}_{31}^2}\bar{a}_{R31} \; , \; \bar{F}_{41} = \frac{Q_1 Q_4}{4\pi\varepsilon_0 \bar{R}_{41}^2}\bar{a}_{R41}$$

$$\bar{R}_{21} = (0-0)\bar{a}_x + (0-0.02)\bar{a}_y + (0-0)\bar{a}_z = -0.02\bar{a}_y$$

$$|\bar{R}_{21}| = 0.02$$

$$\bar{a}_{R21} = \frac{\bar{R}_{21}}{|\bar{R}_{21}|} = \frac{-0.02\bar{a}_y}{0.02} = -\bar{a}_y$$

$$\bar{R}_{31} = (0-0.05)\bar{a}_x + (0-0.02)\bar{a}_y + (0-0)\bar{a}_z = -0.05\bar{a}_x - 0.02\bar{a}_y$$

$$\bar{R}_{31} = \sqrt{0.05^2 + 0.02^2} = 0.0538$$

$$\boxed{\bar{F}_{31} = -0.2597\bar{a}_x - 0.1038\bar{a}_y \qquad \bar{F}_{41} = -0.2157\bar{a}_x}$$

$$\bar{a}_{R31} = \frac{\bar{R}_{31}}{|\bar{R}_{31}|} = \frac{-0.05\bar{a}_x - 0.02\bar{a}_y}{0.0538}$$

$$\bar{R}_{41} = (0-0.05)\bar{a}_x + (0-0)\bar{a}_y + (0-0)\bar{a}_z = -0.05\bar{a}_x$$

$$\bar{R}_{41} = 0.05$$

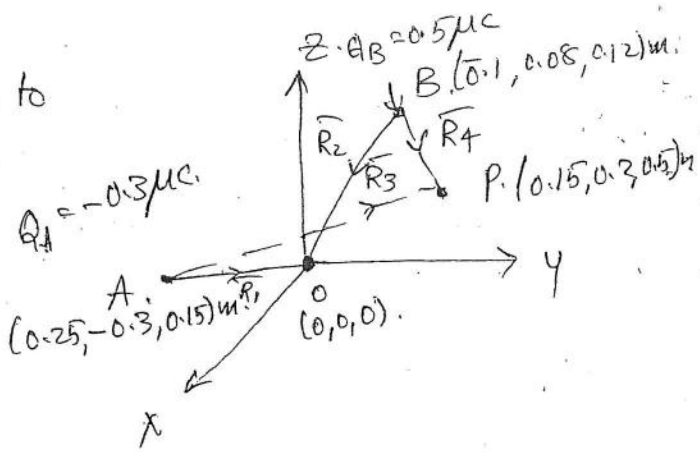
$$\bar{a}_{R41} = \frac{\bar{R}_{41}}{|\bar{R}_{41}|} = \frac{-0.05}{0.05}\bar{a}_x = -\bar{a}_x$$

$$\therefore \bar{F} = -0.4754\bar{a}_x - 1.45189\bar{a}_y \text{ N.}$$

$$\therefore \bar{F}_{21} = \frac{0.3\times10^{-6}\times0.2\times10^{-6}}{4\pi\varepsilon_0(0.0538)^2}[-\bar{a}_y] = \boxed{-1.348\bar{a}_y}$$

1. (2)

i) $\bar{E}$ at the origin $O(0,0,0)$ is due to charges at A & B




$\therefore \bar{E} = \bar{E}_A + \bar{E}_B$

$$= \frac{Q_A}{4\pi\varepsilon_0 R_1^2}\bar{a}_{R1} + \frac{Q_B}{4\pi\varepsilon_0 R_2^2}\bar{a}_{R2}$$

$$\bar{R}_1 = (0-0.25)\bar{a}_x + (0-(-0.3))\bar{a}_y + (0-0.15)\bar{a}_z$$

$$= -0.25\bar{a}_x + 0.3\bar{a}_y - 0.15\bar{a}_z$$

$$|\bar{R}_1| = \sqrt{0.25^2 + 0.3^2 + 0.15^2} = 0.41833m$$

SE(92, 93, A2, 93)
71/57, DS

$$\bar{R}_2 = (0-(-0.1))\bar{a}_x + (0-0.08)\bar{a}_y + (0-0.12)\bar{a}_z$$

$$= 0.1\bar{a}_x - 0.08\bar{a}_y - 0.12\bar{a}_z$$

$$|\bar{R}_2| = \sqrt{0.1^2 + 0.08^2 + 0.12^2} = 0.1755m$$

$$\bar{E}_A = \frac{-0.3\times10^{-6}}{4\pi\times8.854\times10^{-12}\times(0.41833)^2}\left(\frac{-0.25\bar{a}_x + 0.3\bar{a}_y - 0.15\bar{a}_z}{0.41833}\right)$$

$$\bar{E}_B = \frac{0.5\times10^{-6}}{4\pi\times8.854\times10^{-12}\times(0.1755)^2}\left(\frac{0.1\bar{a}_x - 0.08\bar{a}_y - 0.12\bar{a}_z}{0.1755}\right)$$

$$\therefore \bar{E} = 92.344\bar{a}_x - 77.558\bar{a}_y - 94.238\bar{a}_z \text{ KV/m}$$

ii) $\bar{E}$ at point $P(0.15, 0.2, 0.5)$ is due to charges at A & B.

$$\bar{E} = \bar{E}_A + \bar{E}_B = \frac{Q_A}{4\pi\varepsilon_0 R_3^2}\bar{a}_{R3} + \frac{Q_B}{4\pi\varepsilon_0 R_4^2}\bar{a}_{R4}$$

$$\bar{R}_3 = (0.15-0.25)\bar{a}_x + (0.2-(-0.3))\bar{a}_y + (0.5-0.15)\bar{a}_z = -0.1\bar{a}_x + 0.5\bar{a}_y + 0.35\bar{a}_z$$

$$|\bar{R}_3| = 0.6184m.$$

$$\bar{R}_4 = (0.15-(-0.1))\bar{a}_x + (0.2-0.08)\bar{a}_y + (0.5-0.12)\bar{a}_z = 0.25\bar{a}_x + 0.12\bar{a}_y + 0.38\bar{a}_z$$

$$|\bar{R}_4| = 0.4704m.$$

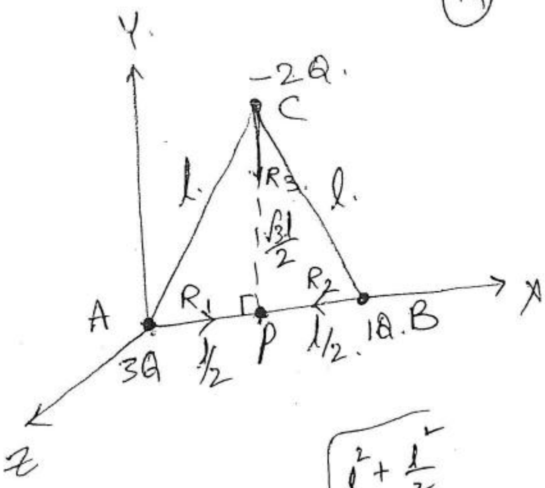
$$\bar{E} = 11.9335\bar{a}_x - 0.52\bar{a}_y + 12.4154\bar{a}_z \text{ KV/m}$$

point charges are located at each corner of an equilateral $\Delta^{le}$. If the charges are 3Q, -2Q & 1Q, find $\bar{E}$ at midpoint of 3Q & 1Q side.

(14)

Let $AB = BC = CA = l$.

$$CP = \frac{\sqrt{3}l}{2}$$




A(0,0,0),

B(l,0,0),

C($l/2$, $\frac{\sqrt{3}l}{2}$, 0)

P($l/2$, 0, 0)

$\bar{E}$ at P is

$$\bar{E}_1 = \frac{Q_1}{4\pi\varepsilon_0 R_1^2}\bar{a}_{R_1} \qquad \bar{r}_p - \bar{r}_A$$

$$\sqrt{l^2 + \frac{l^2}{2}}$$

$$\sqrt{l^2(1+\frac{1}{2})}$$

$$= \sqrt{l^2\frac{3}{2}}$$

$$\bar{R}_1 = \left[\left(\frac{l}{2} - 0\right)\bar{a}_x + 0\bar{a}_y + 0\bar{a}_z\right] = 0.5l\,\bar{a}_x$$

$$|\bar{R}_1| = 0.5l.$$

$$\bar{a}_{R_1} = \frac{\bar{R}_1}{|\bar{R}_1|} = \frac{\cancel{0.5l}\bar{a}_x}{\cancel{0.5l}} = \bar{a}_x \qquad = \bar{r}_{d} - \bar{r}_1$$

$$\therefore \bar{E}_1 = \frac{3Q}{4\pi\varepsilon_0(0.5l)^2}\bar{a}_x = \frac{1.078\times10^{11}Q}{l^2}\bar{a}_x$$

$$\bar{E}_2 = \frac{Q_2}{4\pi\varepsilon_0 R_2}\bar{a}_{R_2} \qquad \bar{r}_p - \bar{r}_B$$

$$\bar{R}_2 = \left(\frac{l}{2} - l\right)\bar{a}_x + 0\bar{a}_y + 0\bar{a}_z = -0.5l\bar{a}_x$$

$$\bar{a}_{R_2} = -\bar{a}_x$$

$$\bar{E}_2 = \frac{1Q}{4\pi\varepsilon_0(0.5l)^2}(-\bar{a}_x) = \frac{-3.595\times10^{10}Q}{l^2}\bar{a}_x$$

$$\bar{E} \text{ at } P = \bar{E}_1 + \bar{E}_2 + \bar{E}_3$$

$$= \frac{Q}{l^2}\left[7.185\times10^{10}\bar{a}_x + 2.3968\times10^{10}\bar{a}_y\right] V/m$$

$$\bar{E}_3 = \frac{Q_3}{4\pi\varepsilon_0 R_3}\bar{a}_{R_3}$$

$$\bar{R}_3 = \bar{r}_p - \bar{r}_c = \left[\left(\frac{l}{2} - \frac{l}{2}\right)\bar{a}_x + \left(0 - \frac{\sqrt{3}l}{2}\right)\bar{a}_y + (0-0)\bar{a}_z\right]$$

$$= -0.866l\,\bar{a}_y$$

$$\bar{a}_{R_3} = -\bar{a}_y$$

$$\therefore \bar{E}_3 = \frac{-2Q}{4\pi\varepsilon_0(0.866l)^2}(-\bar{a}_y) = \frac{2.3968\times10^{10}Q}{l^2}\bar{a}_y$$

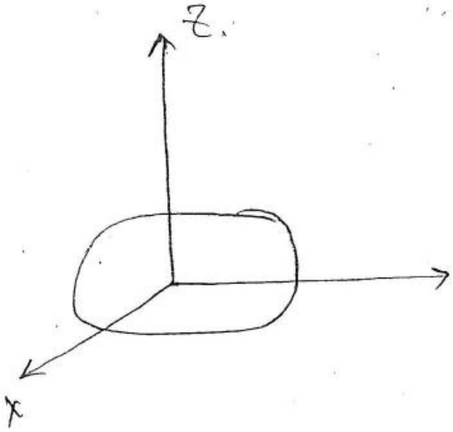
A circular disc of 10 cm radius is charged uniformly with a total charge Find E at a point 20 cm on its axis.

Sol) $Q = 100\,\mu C$.

$\rho = 10\,cm = 0.1\,m$

$\text{area} = \pi\rho^2 = 0.03141\,m^2$

$$\rho_s = \frac{Q}{\text{area}} = \frac{100 \times 10^{-6}}{0.03141} = 3.1831 \times 10^{-3}\,C/m^2$$




$ds = \rho\, d\rho\, d\phi$

$\bar{a}_\rho = -\rho\bar{a}_\rho + z\bar{a}_z$

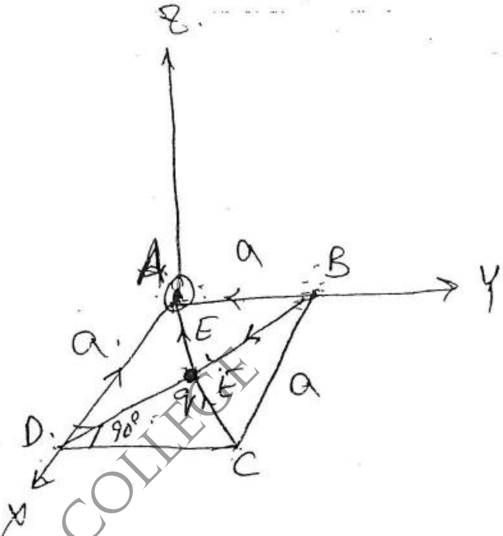
$$\bar{a}_\rho = \frac{-\rho\bar{a}_\rho + z\bar{a}_z}{\sqrt{\rho^2 + z^2}}$$

$$\bar{E} = \int_s \frac{dQ}{4\pi\varepsilon_0 R^2}\bar{a}_\rho$$

$$= \int_0^{2\pi}\int_{\rho=0}^{0.1} \frac{\rho_s(\rho\, d\rho\, d\phi)}{4\pi\varepsilon_0(\rho^2+z^2)} \cdot \frac{z\bar{a}_z}{\sqrt{\rho^2+z^2}}$$

Q. It is required to hold 4 equal point charges each in equilibrium at the corners of a square. Find the point charge which will do this, if placed at the centroid of the square. (15)

Sol: Let the sides of square be of length 'a' and each point charge is of magnitude $Q$ coulombs.




Let corners are A, B, C & D.

E is the centroid of the square.

Let point charge 'q' is placed at E in order to hold the 4 charges in equi.

Calculate the force exerted on a charge at A placed at origin, due to all the charges.

$A(0,0,0)$, $B(0,a,0)$, $C(a,a,0)$, $D(a,0,0)$

$\bar{B} = a\bar{a}_y$, $\bar{C} = a\bar{a}_x + a\bar{a}_y$, $\bar{D} = a\bar{a}_x$

while point E is at $\left(\frac{a}{2}, \frac{a}{2}, 0\right)$

$$\bar{E} = 0.5a\bar{a}_x + 0.5a\bar{a}_y$$

$$\bar{F}_T = \bar{F}_B + \bar{F}_C + \bar{F}_D + \bar{F}_E$$

$$= \frac{QQ}{4\pi\varepsilon_0 R_{BA}^2}\bar{a}_{BA} + \frac{QQ}{4\pi\varepsilon_0 R_{CA}^2}\bar{a}_{CA} + \frac{QQ}{4\pi\varepsilon_0 R_{DA}^2}\bar{a}_{DA} + \frac{Qq}{4\pi\varepsilon_0 R_{EA}^2}\bar{a}_{EA}$$

$$= \frac{Q^2}{4\pi\varepsilon_0}\left[\frac{1}{R_{BA}^2}\frac{\bar{A}-\bar{B}}{|\bar{A}-\bar{B}|} + \frac{1}{R_{CA}^2}\frac{\bar{A}-\bar{C}}{|\bar{A}-\bar{C}|} + \frac{1}{R_{DA}^2}\frac{\bar{A}-\bar{D}}{|\bar{A}-\bar{D}|}\right] + \frac{Qq}{4\pi\varepsilon_0 R_{EA}^2}\frac{\bar{A}-\bar{E}}{|\bar{A}-\bar{E}|}$$

$\bar{A} = (0\bar{a}_x + 0\bar{a}_y + 0\bar{a}_z)$

$\bar{B} = (0\bar{a}_x + a\bar{a}_y + 0\bar{a}_z)$

$\bar{A}-\bar{B} = (0-0) + (0-a) + (0-0)$

$\bar{A}-\bar{B} = -a\bar{a}_y \quad R_{BA} = a$

$\bar{A}-\bar{C} = -a\bar{a}_x - a\bar{a}_y \quad R_{CA} = \sqrt{2}\,a$

$\bar{A}-\bar{D} = -a\bar{a}_x \quad R_{DA} = a$

$\bar{A}-\bar{E} = -0.5a\bar{a}_x - 0.5a\bar{a}_y \quad R_{EA} = \boxed{\sqrt{0.5}}\,a$

$\sqrt{0.25a^2 + 0.25a^2} = \sqrt{(0.5a)^2 + (0.5a)^2} = \sqrt{2(0.5a)^2} = \boxed{\sqrt{2}(0.5a)}$

$\frac{\sqrt{2}}{2} = \frac{\sqrt{1}}{\sqrt{2}} = \sqrt{\frac{1}{2}}$

$$\bar{F}_T = \frac{Q}{4\pi\varepsilon_0 a^2}\left[\left(-1.3535Q - \frac{q}{\sqrt{0.5}}\right)\bar{a}_x + \left[-1.3535Q - \frac{q}{\sqrt{0.5}}\right)\bar{a}_y\right]$$

To hold all the charges in equilibrium, the net Force exerted on any of the charges due to other charges must be zero ie, $\bar{F}_T = 0$

but $\frac{Q}{4\pi\varepsilon_0 a^2} \neq 0$.

$$\therefore\ -1.3535Q - \frac{q}{\sqrt{0.5}} = 0$$

$$q = -1.3535 \times \sqrt{0.5}\ Q$$

$$= -0.9571\ Q \text{ coulo}$$

This charge is required at the centroid to hold all the charges in equilibrium

Thus if $Q = 1\ \mu C$ then $q = -0.9571\ \mu C$.

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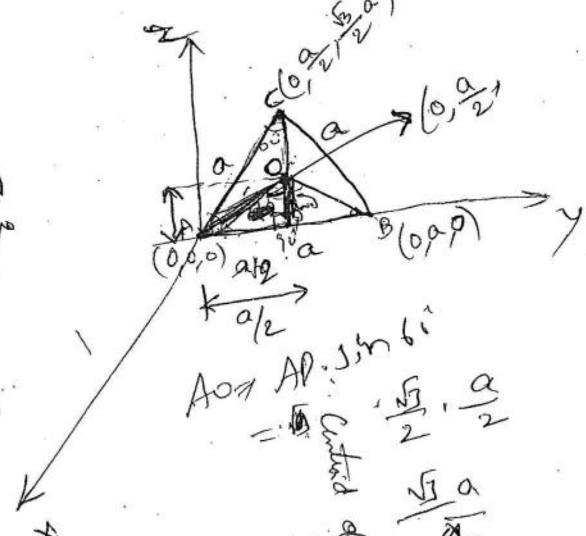
* It is required to hold the 3 point charges $+Q$ each in equilibrium at the corners of an equilateral $\Delta^{le}$ find the point charge which will do this if placed at the centroid of a $\Delta^{le}$.




$AO = AP \cdot \sin 60°$

$= \frac{\sqrt{3}}{2} \cdot \frac{a}{2}$

$\frac{\sqrt{3}\,a}{4}$

Centroid of equi.

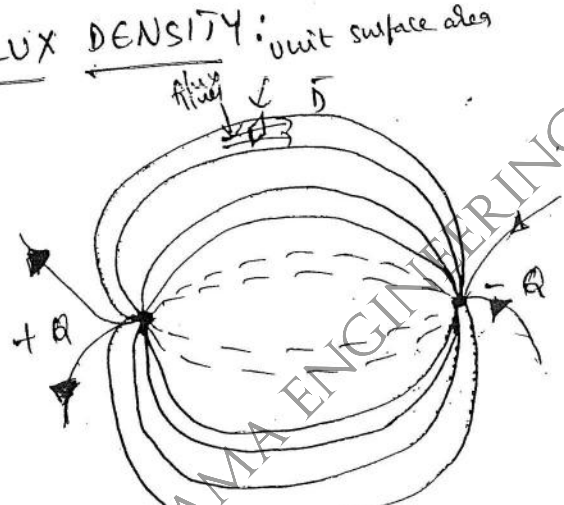
ECTRIC FLUX:

The total no. of lines of force in any particular electric field is called the electric flux.

Properties:

1) start from +ve & terminate -ve
2) -ve charge is absent then the flux terminates at infinity
3) more no. of lines, electric field is stronger
4) lines are parallel never cross
5) lines are independent of medium
6) lines always enter or leave the charged surface, normally

ELECTRIC FLUX DENSITY:




Consider 2 point charges as shown in fig, the flux lines originating from +ve & terminating at -ve are shown in the form of tubes.

Consider a unit surface area as shown in fig.

The net flux passing normal through the unit surface area is called the electric flux density, denoted by $\bar{D}$

$$D = \frac{\psi}{S}$$

$\psi$ = total flux

S = total surface area

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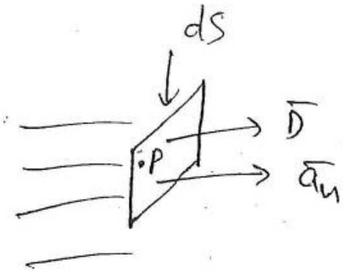
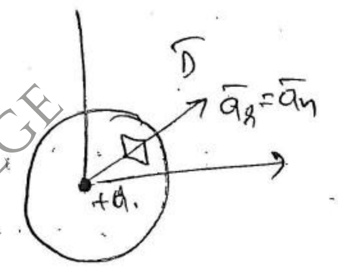
Vector form:

$$\bar{D} = \frac{d\psi}{dS}\,\bar{a}_n \ C/m^2$$




$\bar{D}$ due to a point charge:

- Consider a point charge $+Q$ placed at the centre of the imaginary sphere of radius $r$.
- flux lines originating from the point charge $+Q$ are directed radially outwards.




$$\bar{D} = \frac{\text{total flux } \psi}{\text{Total area } S}$$

$$|\bar{D}| = \frac{Q}{4\pi r^2}\,\bar{a}_r \ C/m^2 \rightarrow (1)$$

RELATIONSHIP b/w $\bar{D}$ & $\bar{E}$:

$$\bar{E} = \frac{Q}{4\pi\varepsilon r^2}\,\bar{a}_r \rightarrow (2)$$

from (1) & (2)

$$\frac{\bar{D}}{\bar{E}} = \frac{\frac{Q}{4\pi r^2}\bar{a}_r}{\frac{Q}{4\pi\varepsilon r^2}\bar{a}_r} = \varepsilon$$

$$\boxed{\therefore \bar{D} = \varepsilon\bar{E}}$$

(1) $\bar{D}$ & $\bar{E}$, both act in same direction
(2) related with permittivity of the medium
(3)

due to a line charge:

$$Q = \int_L \rho_L \, dL$$

$$\bar{D} = \frac{Q}{4\pi r^2}\bar{a}_r = \frac{\int_L \rho_L \, dl}{4\pi r^2}\bar{a}_r$$

if line charge is infinite;

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon r}\bar{a}_r$$

$$\bar{D} = \frac{\rho_L}{2\pi r}\bar{a}_r$$

due to Surface charge:

$$Q = \int_S \rho_s \, ds$$

$$\bar{D} = \frac{\int_s \rho_s \, ds}{4\pi r^2}\bar{a}_r$$

if the sheet of charge is infinite

$$\bar{E} = \frac{\rho_s}{2\varepsilon}\bar{a}_n$$

$$\bar{D} = \frac{\rho_s}{2}\bar{a}_n$$

due to Volume charge:

$$Q = \int_V \rho_v \, dv$$

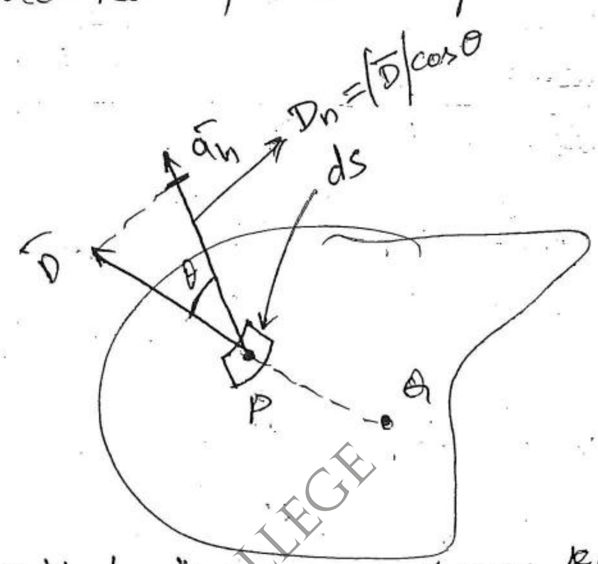
$$\bar{D} = \frac{\int_{vol} \rho_v \, dv}{4\pi r^2}\bar{a}_r$$

GAUSS LAW:

The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

MATHEMATICAL REPRESENTATION:

Consider any object of irregular shape as shown in fig.
The total charge enclosed by the irregular closed surface is Q Coulombs.
Consider a small differential surface ds at point P.
As the surface is irregular, the




direction of $\bar{D}$ as well as its magnitude is going to change from point to point on the surface.
The surface ds under consideration can be represented in the vector form in terms of its area & direction normal to the surface at the point.

$$d\bar{s} = ds\,\bar{a}_n$$

→ flux density at point P is $\bar{D}$ & its direction is such that it makes an angle $\theta$ with the normal direction at point P.

→ flux $d\psi$ passing through the surface dS is the product of the component of $\bar{D}$ in the direction normal to the dS & $d\psi$

$$d\psi = D_n\, dS$$

$$D_n = |\bar{D}| \cos\theta$$

$$d\psi = |\bar{D}| \cos\theta\, ds$$

From dot product

$$|\bar{D}|\, |d\bar{s}| \cos\theta = \bar{D} \cdot d\bar{s}$$

$$\therefore\ d\psi = \bar{D} \cdot d\bar{s}$$

This is the flux passing through incremental surface area dS. Hence the total flux passing through the entire closed surface is to be obtained by finding the surface integration (17)

$$\Psi = \int d\Psi = \oint_S \bar{D} \cdot d\bar{S}$$

$\oint_S$ sign indicates the integration over the closed surface & called closed surface integral.

Such a closed surface over which the integration is carried out is called Gaussian surface

$$\boxed{\therefore \ \Psi = \oint_S \bar{D} \cdot d\bar{S} = Q}$$

(or)

The surface over which Gauss law is applied is called Gaussian surface.

1) The surface may be irregular but should be sufficiently large so as to enclose the entire charge.
2) The surface must be closed.
3) At each point of the surface $\bar{D}$ is either normal or tangential to the surface.
4) The electric flux density D is constant over the surface at which $\bar{D}$ is normal.

APPLICATIONS OF GAUSS LAW:

Gauss law can be used to find $\bar{E}$ or $\bar{D}$ for symmetrical charge distributions.

→ used to find the charge enclosed or the flux passing through the closed surface.

→ cannot be used for charge distribution if not symmetrical.

Conditions to be satisfied while selecting the closed Gaussian surface.

(1) $\bar{D}$ is everywhere either normal or tangential to the closed surface i.e., $\theta = \pi/2$ or $\pi$

(2) $\bar{D}$ is constant over the portion of the closed surface for which $\bar{D} \cdot d\bar{s}$ is not zero.

PROOF OF GAUSS LAW (point charge):

→ Let a point charge Q is located at the origin. to determine $\bar{D}$ & to apply Gauss's law, consider a spherical surface around Q, with centre as origin.

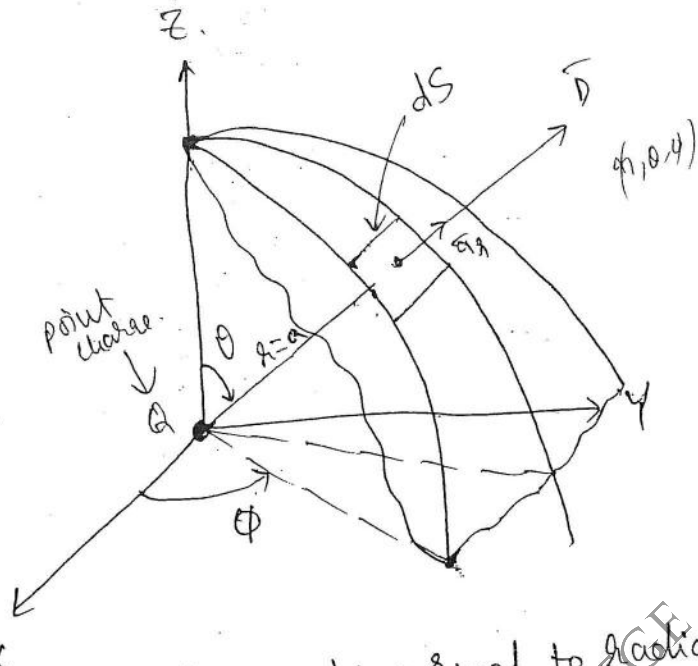
→ This spherical surface is gaussian surface & it satisfies required condition.

→ $\bar{D}$ is always directed radially outwards along $\bar{a}_r$ which is normal to the spherical surface at any point P on the surface.

→ Consider a differential surface area ds as shown. The direction normal to the surface ds is $\bar{a}_r$, considering spherical coordinate system.

→ Radius of the sphere is $r = a$

→ the direction of $\bar{D}$ is along $\bar{a}_r$ which is normal to ds at any point P




69, 70, 77, 80, 81, A2, A3, A9, B1, B3, C3

In spherical coordinate system, ds normal to radial direction $\bar{a}_r$ is

$$ds = r^2 \sin\theta \, d\theta \, d\phi = a^2 \sin\theta \, d\theta \, d\phi$$

$$d\bar{s} = ds \, \bar{a}_n = a^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r \rightarrow (1)$$

$$\bar{D} = \frac{Q}{4\pi r^2} \bar{a}_r \Rightarrow \frac{Q}{4\pi a^2} \bar{a}_r \rightarrow (2)$$

$$\bar{D} \cdot d\bar{s} = |\bar{D}| \, |d\bar{s}| \cos\theta'$$

$$|\bar{D}| = \frac{Q}{4\pi a^2} ; \; |d\bar{s}| = a^2 \sin\theta \, d\theta \, d\phi , \; \theta' = 0^\circ$$

the normal to $d\bar{s}$ is $\bar{a}_r$, while $\bar{D}$ also acts along $\bar{a}_r$

hence $d\bar{s}$ & $\bar{D}$ angle is $0^\circ$.

$$\bar{D} \cdot d\bar{s} = \frac{Q}{4\pi a^2} a^2 \sin\theta \, d\theta \, d\phi \cos 0^\circ$$

$$= \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

$$\bar{D}\cdot d\bar{S} = \frac{Q}{4\pi a^2}\,\bar{a_r}\cdot a^2 \sin\theta\, d\theta\, d\phi\, \bar{a_r}$$

$$= \frac{Q}{4\pi}\sin\theta\, d\theta\, d\phi\,[\bar{a_r}\cdot\bar{a_r}]$$

$$\bar{D}\cdot d\bar{S} = \frac{Q}{4\pi}\sin\theta\, d\theta\, d\phi$$

$$\Psi = \oint_S \bar{D}\cdot d\bar{S}$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{Q}{4\pi}\sin\theta\, d\theta\, d\phi$$

$$= \frac{Q}{4\pi}\left[-\cos\theta\right]_0^{\pi}\left[\phi\right]_0^{2\pi}$$

$$= \frac{Q}{\cancel{4\pi}}\left(-(-1)-(-1)\right)(2\pi)$$

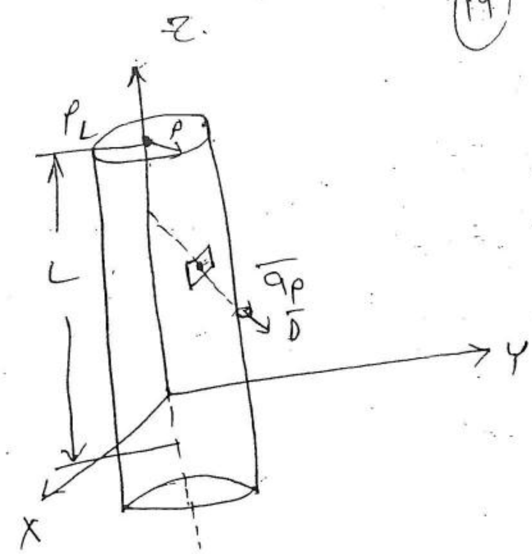
$$= Q$$

$$\boxed{\therefore\ \Psi = Q}$$

INFINITE LINE CHARGE:

Consider an infinite line charge of density $\rho_L$ c/m lying along z-axis from $-\infty$ to $+\infty$

Consider the Gaussian surface as the right circular cylinder with z-axis as its axis & radius $\rho$, the length of the cylinder is L




As the line charge is along z-axis there cannot be any component of $\bar{D}$ in z direction.

$$Q = \oint_s \bar{D} \cdot \bar{dS}$$

The integration is to be evaluated for side surface, top & bottom.

$$Q = \oint_{side} \bar{D} \cdot \bar{dS} + \oint_{top} \bar{D} \cdot \bar{dS} + \oint_{bottom} \bar{D} \cdot \bar{dS}$$

$\because$ $\bar{D}$ has no component on +z & -z

$$\oint_{top} \bar{D} \cdot \bar{dS} = \oint_{bottom} \bar{D} \cdot \bar{dS} = 0$$

$$\therefore \oint_{side} \bar{D} \cdot \bar{dS} \Rightarrow \quad \bar{D} = D_\rho \bar{a}_\rho$$

$$\bar{dS} = \rho \, d\phi \, dz \, \bar{a}_\rho$$

$$\bar{D} \cdot \bar{dS} = D_\rho \rho \, d\phi \, dz \, (\bar{a}_\rho \cdot \bar{a}_\rho)$$

$$\oint_{side} \bar{D} \cdot \bar{dS} = D_\rho \rho \, d\phi \, dz$$

$$\Rightarrow \; = \int_{z=0}^{L} \int_{\phi=0}^{2\pi} D_\rho \rho \, d\phi \, dz$$

$$= \rho D_\rho [z]_0^L [\phi]_0^{2\pi}$$

$$\oint_{side} \bar{D}\cdot d\bar{s} = 2\pi\rho D_\rho L$$

$$Q = 2\pi\rho D_\rho L$$

$$D_\rho = \frac{Q}{2\pi\rho L}$$

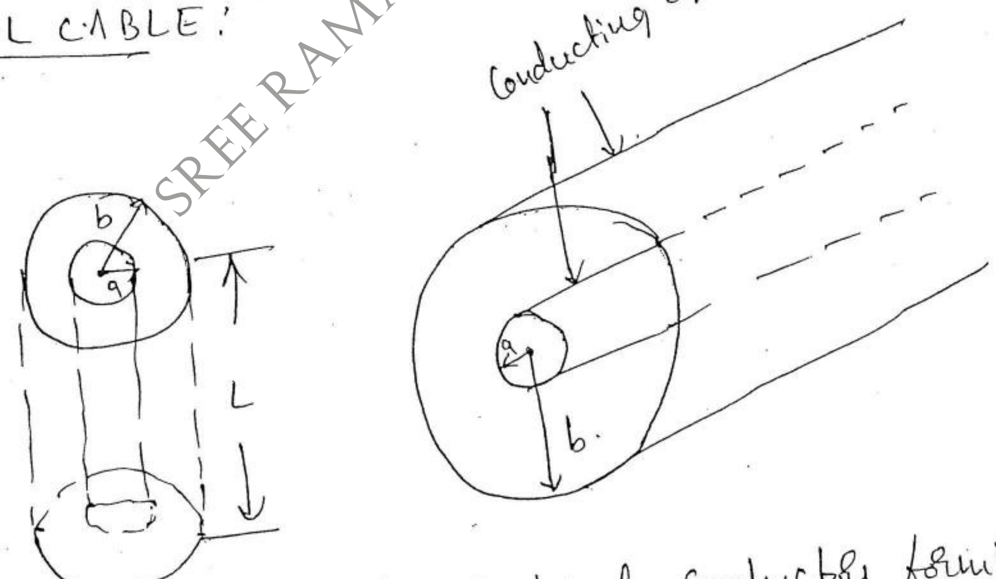
$$\bar{D} = D_\rho \bar{a}_\rho = \frac{Q}{2\pi\rho L}\bar{a}_\rho$$

$$\frac{Q}{L} = \rho_L \ \text{c/m}$$

$$\boxed{\bar{D} \Rightarrow \frac{\rho_L}{2\pi\rho}\bar{a}_\rho \ \text{c/m}^2}$$

$$\mathcal{E}_1 \ \bar{E} = \frac{\bar{D}}{\mathcal{E}} = \frac{\rho_L}{2\pi\mathcal{E}\rho}\bar{a}_\rho \ \text{c/m}^2$$

CO-AXIAL CABLE:




Consider the 2 co-axial cylindrical conductors forming a coaxial cable. The radius of inner conductor is 'a' while the radius of the outer conductor is 'b', the length of the cable is L.

The charge distribution on the outer surface of the inner conductor is having density $\rho_S$ $C/m^2$. (20)

The total outer surface area of the inner conductor is $2\pi a L$.

hence $\rho_S$ can be expressed in terms of $\rho_L$

$\rho_S = \frac{Q}{S}$

$= \frac{Q}{L \times L}$

$\rho_S = \frac{\rho_L}{L}$

$\rho_S \times L = \rho_L$

$\frac{\rho_S \times L \times L}{L} = \rho_L$

$$\boxed{\rho_L = \frac{\rho_S \times \text{surface area}}{\text{Total length}} = \frac{\rho_S \times 2\pi a L}{L}}$$

$\rho_L = \rho_S L$

$$\rho_L = 2\pi a \rho_S \; C/m$$

Thus the line charge density of inner conductor is $\rho_L$ C/m.

Consider the right circular cylinder of length L as the Gaussian surface.

$$Q = D_\rho 2\pi \rho L \;\; (\text{From infinite line charge})$$

$\underline{a < \rho < b}$

The total charge on the inner conductor is to be obtained by evaluating the surface integral of the surface charge distribution

$$Q = \oint_S \rho_S \, d\bar{S}$$

$ds = \rho \, d\phi \, dz$

$= a \, d\phi \, dz$

$$Q = \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho_S \, a \, d\phi \, dz$$

$Q = 2\pi a \cancel{L} \rho_S$

$Q = D_\rho 2\pi \cancel{\rho} \cancel{L}$

$\Rightarrow D_\rho = \frac{a \rho_S}{\rho} \bar{a}_\rho$

$\bar{D} = \frac{a\rho_S}{\rho} \bar{a}_\rho$

$\rho_S = \frac{\rho_L}{2\pi a}$

$\bar{D} = \frac{a \rho_L}{2\pi a \rho} \bar{a}_\rho$

$$\boxed{\bar{D} = \frac{\rho_L}{2\pi \rho} \bar{a}_\rho \; C/m^2}$$

$\bar{E} = \frac{\rho_L}{2\pi \varepsilon \rho} \bar{a}_\rho \;\; \underline{(a < \rho < b)} \; V/m$

$\rho > b$

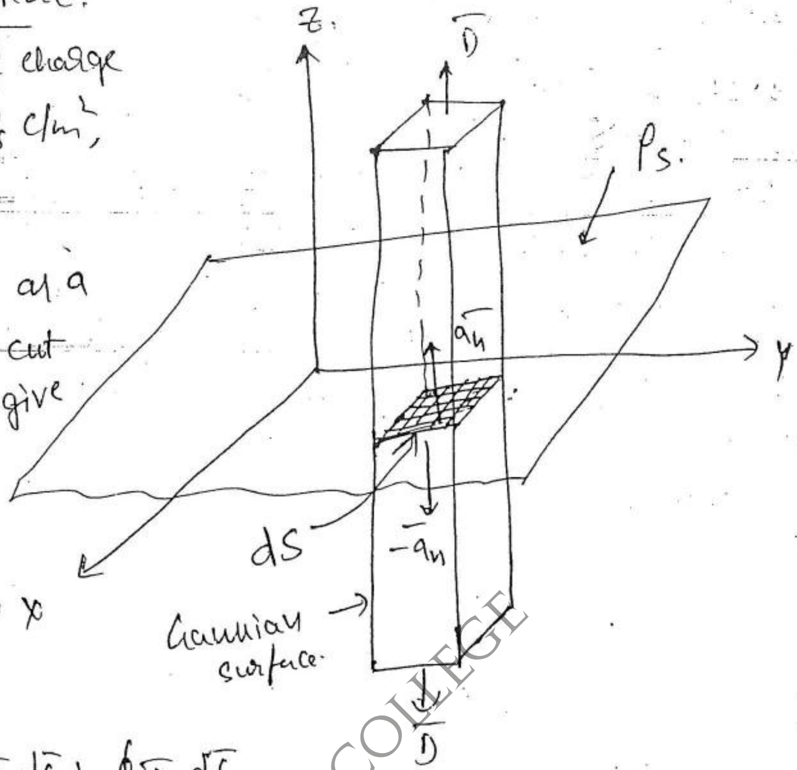
## INFINITE SHEET OF CHARGE:

Consider the infinite sheet of charge of uniform charge density $\rho_s$ $C/m^2$, lying in $Z=0$ plane.

Consider a rectangular box as a Gaussian surface which is cut by the sheet of charge to give

$ds = dx\,dy$

hence $\bar{D}=0$ in $x$ & $y$ direction $x$




$$Q = \oint_S \bar{D}\cdot d\bar{s} = \oint_{sides} \bar{D}\cdot d\bar{s} + \oint_{top} \bar{D}\cdot d\bar{s} + \oint_{bottom} \bar{D}\cdot d\bar{s}$$

but $\oint_{sides} \bar{D}\cdot d\bar{s} = 0$ as $\bar{D}$ has no components in $x$ & $y$ directions

Now $\oint_{top} \bar{D}\cdot d\bar{s} \Rightarrow \bar{D} = D_z \bar{a_z}$

$$d\bar{s} = dx\,dy\,\bar{a_z}$$

$$\bar{D}\cdot d\bar{s} = D_z\,dx\,dy\,(\bar{a_z}\cdot\bar{a_z})$$

$$= D_z\,dx\,dy$$

$\oint_{bottom} \bar{D}\cdot d\bar{s} \Rightarrow \bar{D} = -D_z(-\bar{a_z})$

$$d\bar{s} = dx\,dy\,(-\bar{a_z})$$

$$\bar{D}\cdot d\bar{s} = D_z\,dx\,dy\,(\bar{a_z}\cdot\bar{a_z})$$

$$= D_z\,dx\,dy$$

& $\oint_{top} dx\,dy = \oint_{bottom} dx\,dy = A =$ area of surface

$$Q = 2D_z A$$

$$\frac{Q}{A} = \rho_s$$

$$\rho_s = 2D_z$$

$$D_z = \frac{\rho_s}{2}$$

$$\bar{D} = D_z \bar{a_z} = \frac{\rho_s}{2} \bar{a_z} \ c/m^2$$

$$\bar{E} = \frac{\rho_s}{2\varepsilon} \bar{a_z} \ V/m$$

GAUSS'S LAW Applied to Differential Volume Element:

→ Until now we have considered various cases in which there exists a symmetry & component of $\bar{D}$ is normal to the surface & constant everywhere on the surface. But if there does not exist a symmetry & Gaussian surface can not be chosen such that normal component of $\bar{D}$ is constant or zero everywhere on the surface, Gauss's law can not be directly applied.

→ In such a case a differential closed Gaussian surface is considered. The closed surface is so small that $\bar{D}$ is almost const. everywhere on the surface. Finally results can be obtained by decreasing the volume enclosed by Gaussian surface to approach to zero.

→ Consider a Cartesian-coordinate system & a point P in it such that the electric flux density at P is given by,

$$\bar{D} = D_x \bar{a_x} + D_y \bar{a_y} + D_z \bar{a_z}$$

→ Consider the closed Gaussian differential surface in the form of rectangular box, which is a differential volume element.

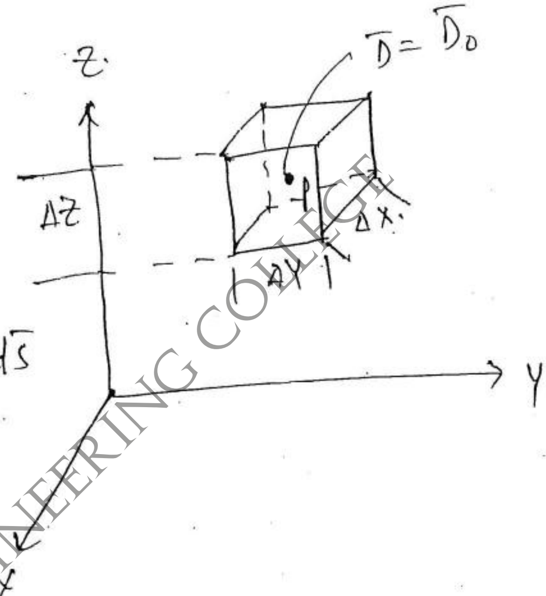
the sides of this element are $\Delta x, \Delta y$ & $\Delta z$.

, The position of this element is such that the point P is at the centre of the element & treated to be origin.

, Hence $\bar{D}$ at P is $\bar{D_0}$

$$\bar{D} = \bar{D_0} = D_{x0}\bar{a_x} + D_{y0}\bar{a_y} + D_{z0}\bar{a_z}$$

the component $D_{x0}, D_{y0}$ & $D_{z0}$ vary with distance in the respective directions.




$\Rightarrow \quad Q = \oint_S \bar{D}.d\bar{s}$

ie total surface integral is

$$\oint_S \bar{D}.d\bar{s} = \left[\int_f + \int_b + \int_L + \int_R + \int_T + \int_B\right]\bar{D}.d\bar{s}$$



FRONT SURFACE:

$\int_{front} \bar{D}.d\bar{s} \Rightarrow d\bar{s} = \Delta y \Delta z\, \bar{a_x}$

$\bar{D} = \bar{D}_{front}$

$\bar{D}_{front} = D_{x,front}\, \bar{a_x}$

$$\therefore \int_{front} \bar{D}.d\bar{s} = D_{x,front}\, \Delta y \Delta z\, [\bar{a_x}.\bar{a_x}]$$

$$D_{x,front} = D_{x0} + \begin{bmatrix} \text{Rate of change} \\ \text{of } D_x \text{ with } x \end{bmatrix} \times \begin{bmatrix} \text{Distance of} \\ \text{surface} \\ \text{from } P \end{bmatrix}$$

$$= D_{x0} + \frac{\partial D_x}{\partial x} \cdot \frac{\Delta x}{2}$$

→ The point P is at the centre so distance of surface in x direction from P is $\frac{\Delta x}{2}$

→ The rate of change is expressed as partial derivative as $D_x$ varies with y & z coordinates also.

$$\int_{front} \bar{D}\cdot d\bar{s} \cong \left[D_{x0} + \frac{\Delta x}{2}\cdot\frac{\partial D_x}{\partial x}\right]\Delta y \Delta z \rightarrow (1)$$

BACK SURFACE:

$$\int_{back} \bar{D}\cdot d\bar{s} \Rightarrow d\bar{s} = \Delta y \Delta z(-\bar{a}_x)$$ (∵ The surface considered from point P is in negative x)

$$\bar{D}_{back} = D_{x\,back}(\bar{a}_x)$$ (∵ flux entering from back side & leaving from front in +ve x direction)

$$\int_{back} \bar{D}\cdot d\bar{s} \cong -D_{x,back}\,\Delta y \Delta z$$

$$D_{x,back} = D_{x0} - \left(\frac{\partial D_x}{\partial x}\right)\left(\frac{\Delta x}{2}\right)$$

$$\therefore \int_{back} \bar{D}\cdot d\bar{s} = -\left[D_{x0} - \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right]\Delta y \Delta z \rightarrow (2)$$

$$\therefore \int_{front} + \int_{back} = 2\,\frac{\Delta x}{2}\cdot\frac{\partial D_x}{\partial x}\,\Delta y \Delta z$$

$$= \frac{\partial D_x}{\partial x}\,\Delta x \Delta y \Delta z$$

IIIly

$$\int_{left} + \int_{right} = \frac{\partial D_y}{\partial y}\,\Delta x \Delta y \Delta z$$

$$\int_{top} + \int_{bottom} = \frac{\partial D_z}{\partial z}\,\Delta x \Delta y \Delta z$$

$$\therefore \oint_s \bar{D}\cdot d\bar{s} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x \Delta y \Delta z$$

$$\Delta x \Delta y \Delta z = \Delta V$$

$\therefore Q$ = charge enclosed in volume $\Delta V = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta V$

DIVERGENCE:

$$Q = \left( \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta V \rightarrow (1)$$

$$Q = \oint_S \bar{D} \cdot d\bar{s} \rightarrow (2)$$

to apply gauss's law we have assumed a differential volume element as the gaussian surface, over which $\bar{D}$ is const. hence eqn (1) & (2) can be equated in limiting case as $\Delta V \rightarrow 0$

$$\oint_S \bar{D} \cdot d\bar{s} = \lim_{\Delta V \to 0} \left( \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta V$$

(or)

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \lim_{\Delta V \to 0} \frac{Q}{\Delta V} = \lim_{\Delta V \to 0} \frac{\oint_S \bar{D} \cdot d\bar{s}}{\Delta V}$$

thus in general if $\bar{A}$ is any vector eg. force, velocity, etc.,

then

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \lim_{\Delta V \to 0} \frac{\oint_S \bar{A} \cdot d\bar{s}}{\Delta V}$$

where mathematical operation on $\bar{A}$ is called a divergence. denoted as $\operatorname{div} \bar{A}$

$$\therefore \operatorname{div} \bar{A} = \lim_{\Delta V \to 0} \frac{\oint_S \bar{A} \cdot d\bar{s}}{\Delta V} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

→ the divergence of a vector is a scalar quantity

$$\rightarrow \quad \nabla = \frac{\partial}{\partial x} \bar{a}_x + \frac{\partial}{\partial y} \bar{a}_y + \frac{\partial}{\partial z} \bar{a}_z$$

$$\rightarrow \quad \nabla \cdot \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \operatorname{div} \bar{A}$$

$\therefore dv = \rho\, d\rho\, d\phi\, dz$

$\therefore dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$.

$$\nabla \cdot \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \cdot \bar{A} = \frac{1}{\rho}\frac{\partial \rho A_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \cdot \bar{A} = \frac{1}{r^2}\frac{\partial r^2 A_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, A_\theta) + \frac{1}{r\sin\theta}\cdot\frac{\partial A_\phi}{\partial \phi}$$

PHYSICAL MEANING OF DIVERGENCE:

$$\rightarrow \quad \operatorname{div} \bar{D} = \lim_{\Delta V \to 0} \frac{\oint_S \bar{D}\cdot d\bar{S}}{\Delta V} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

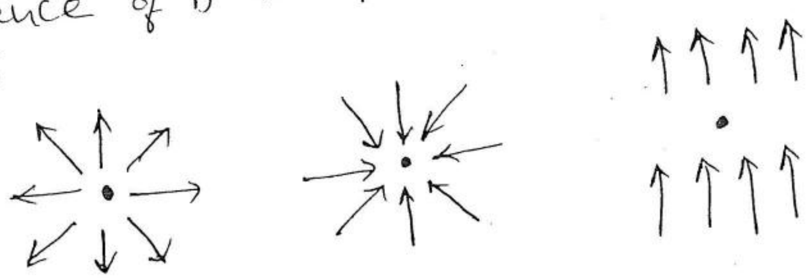
→ The divergence of the vector flux density $\bar{D}$ is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.

→ Hence the divergence of $\bar{D}$ at a given point is a measure of how much the field represented by $\bar{D}$ diverges or converges from that point.

→ If the field is diverging at P then div $\bar{D}$ at point P is +ve

→ " " " spreading out from P then div $\bar{D}$ " is −ve.

→ If the field whatever is converging the same is diverging than the divergence of $\bar{D}$ at point P is zero.



## MAXWELL's FIRST EQUATION:-

→ The divergence of electric flux density $\bar{D}$ is given by ...

$$\text{div } \bar{D} = \lim_{\Delta V \to 0} \frac{\oint_S \bar{D} \cdot d\bar{s}}{\Delta V} \rightarrow (1)$$

$$= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

from gauss law ⇒ $\psi = Q = \oint_S \bar{D} \cdot d\bar{s}$

& expressing gauss law in per unit volume basis

$$\frac{Q}{\Delta V} = \frac{\oint_S \bar{D} \cdot d\bar{s}}{\Delta V}$$

taking limit $\Delta V \to 0$ i.e., volume shrinks to zero

$$\lim_{\Delta V \to 0} \frac{Q}{\Delta V} = \lim_{\Delta V \to 0} \frac{\oint_S \bar{D} \cdot d\bar{s}}{\Delta V}$$

but $\lim_{\Delta V \to 0} \frac{Q}{\Delta V} = \rho_V$ at that point .. → (5)

$$\text{div } \bar{D} = \rho_V$$

$$\boxed{\nabla \cdot \bar{D} = \rho_V}$$

The above equation is called Maxwell's I equation applied to electrostatics also called as point form of Gauss law.

The statement of Gauss law in point form is

The divergence of electric flux density in a medium at a point (differential volume shrinking to zero), is equal to the volume charge density at the same point.

# DIVERGENCE THEOREM:

from gauss law

$$Q = \oint_S \bar{D} \cdot d\bar{S} \rightarrow (1)$$

while the charge enclosed in a volume is given by

$$Q = \int_V \rho_v \, dv$$

according to Gauss law in the point form

$$\nabla \cdot \bar{D} = \rho_v$$

$$Q = \int_V (\nabla \cdot \bar{D}) \, dv \rightarrow (2)$$

from (1) & (2)

$$\boxed{\oint_S \bar{D} \cdot d\bar{S} = \int_V (\nabla \cdot \bar{D}) \, dv}$$

Statement: The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by that closed surface.

1) Find the total charge in a volume defined by the 6 planes for which $1 \le x \le 2$, $2 \le y \le 3$, $3 \le z \le 4$ if,

$$\bar{D} = 4x\bar{a}_x + 3y^2\bar{a}_y + 2z^3\bar{a}_z \ C/m^3$$

$d\bar{s}$ for each surface:

1) front surface $(x=2)$, $ds = dy\,dz \Rightarrow d\bar{s} = dy\,dz\,\bar{a}_x$

2) Back $(x=1)$, $ds = dy\,dz \Rightarrow d\bar{s} = -dy\,dz\,\bar{a}_x$

(3) R-side $(y=3)$, $ds = dx\,dz \Rightarrow d\bar{s} = dx\,dz\,\bar{a}_y$

(4) L-side $(y=2)$, $ds = dx\,dz \Rightarrow d\bar{s} = -dx\,dz\,\bar{a}_y$

(5) top $(z=4)$, $ds = dx\,dy \Rightarrow d\bar{s} = dx\,dy\,\bar{a}_z$

(6) Bottom $(z=3)$, $ds = dx\,dy \Rightarrow d\bar{s} = -dx\,dy\,\bar{a}_z$

$$\bar{D}\cdot d\bar{s} = 4x\,dy\,dz$$
$$= -4x\,dy\,dz$$
$$= 3y^2\,dx\,dz$$
$$= -3y^2\,dx\,dz$$
$$= 2z^3\,dx\,dy$$
$$= -2z^3\,dx\,dy$$

$$\oint_s \bar{D}\cdot d\bar{s} = \int_{z=3}^{4}\int_{y=2}^{3} 4x\,dy\,dz + \int_{z=3}^{4}\int_{y=2}^{3} -4x\,dy\,dz + \int_{z=3}^{4}\int_{x=1}^{2} 3y^2\,dx\,dz + \int_{z=3}^{4}\int_{x=1}^{2} -3y^2\,dx\,dz + \int_{y=2}^{3}\int_{x=1}^{2} 2z^3\,dx\,dz + \int_{y=2}^{3}\int_{x=1}^{2} -2z^3\,dx\,dz$$

$$\boxed{\oint \bar{D}\cdot d\bar{s} = 93C}$$

[or]

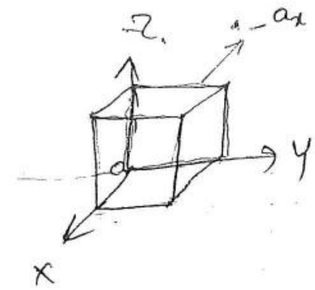
$$\nabla\cdot\bar{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 4 + 6y + 6z^2$$

$$\int_V (\nabla\cdot\bar{D})\,dv = \int_{z=3}^{4}\int_{y=2}^{3}\int_{x=1}^{2} (4 + 6y + 6z^2)\,dx\,dy\,dz$$

$$\boxed{\oint \bar{D}\cdot d\bar{s} = \int_V (\nabla\cdot\bar{D})\,dv = 93C}$$

Determine the net flux of the vector field $D(x, y, z) = 2x^2y\bar{a}_x + z\bar{a}_y + y\bar{a}_z$ emerging from the unit cube $0 \le x, y, z \le 1$ (23)

$\therefore \psi = \oint_S \bar{D} \cdot d\bar{s}$




i) front surface $(x=1)$, $ds = dy\,dz\,\bar{a}_x$
ii) back surface $(x=0)$, $ds = -dy\,dz\,\bar{a}_x$
iii) Right $(y=1)$ $ds = dx\,dz\,\bar{a}_y$
iv) Left $(y=0)$ $ds = -dx\,dz\,\bar{a}_y$
v) Top $(z=1)$ $ds = dx\,dy\,\bar{a}_z$
vi) bottom $(z=0)$ $ds = -dx\,dy\,\bar{a}_z$

$(2x^2y\bar{a}_x + z\bar{a}_y + y\bar{a}_z) \cdot dy\,dz\,\bar{a}_x$

$(2x^2y\,dy\,dz \cdot 1 + 0 + 0)$

for front $\bar{D} \cdot d\bar{s} = 2x^2y\,dy\,dz$, $x=1$

$\bar{D} \cdot d\bar{s} = -2x^2y\,dy\,dz$. $x=0$

right $= z\,dx\,dz$. $y=1$

$= -z\,dx\,dz$ $y=0$

top $= y\,dx\,dy$ $z=1$

$= -y\,dx\,dy$ $z=0$.

$2x^2\left[\frac{y^2}{2}\right]_0^1 [z]_0^1$

$$\oint \bar{D} \cdot d\bar{s} = \int_{z=0}^{1}\int_{y=0}^{1} 2x^2y\,dy\,dz \,\Big|_{x=1} + \int_{z=0}^{1}\int_{y=0}^{1} -2x^2y\,dy\,dz \,\Big|_{x=0}$$

$\nabla \cdot D\,dv$

$$+ \int_{z=0}^{1}\int_{x=0}^{1} z\,dx\,dz \,\Big|_{y=1} + \int_{z=0}^{1}\int_{x=0}^{1} -z\,dx\,dz \,\Big|_{y=0}$$

$\oint D \cdot ds = \nabla \cdot D\,dv$

$\nabla \cdot D$

$$+ \int_{y=0}^{1}\int_{x=0}^{1} y\,dx\,dy + \int_{y=0}^{1}\int_{x=0}^{1} -y\,dx\,dy$$

$$= (2)(1)^2\left(\frac{y^2}{2}\right)_0^1 (z)_0^1 + (0) + \left(\frac{z^2}{2}\right)_0^1 [x]_0^1 + \left(\frac{-z^2}{2}\right)_0^1 (x)_0^1$$

$$+ \left(\frac{y^2}{2}\right)_0^1 (x)_0^1 + \left(\frac{-y^2}{2}\right)_0^1 (x)_0^1.$$

$$= 1 + 0 + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = 1C$$

$$\boxed{\psi = 1C.}$$

# ELECTROMAGNETIC FIELD THEORY
## (20A02403T)

## Energy and Potential

## (ELECTROSTATICS)

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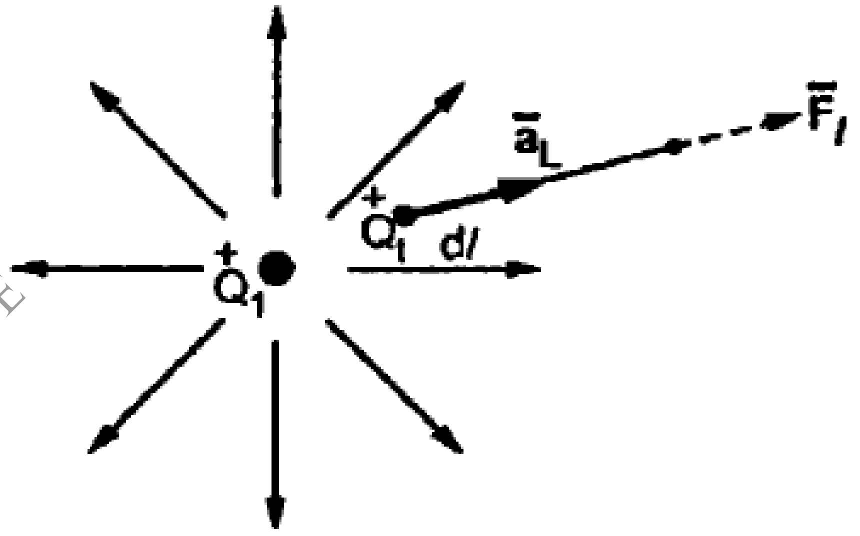
## Work Done

Consider an earth's gravitational field. An object falls on the earth due to the force exerted by earth's gravitational field. But to move an object away from the earth's gravitational field, the work is required to be done by an external source. The force in opposite direction to that exerted by earth's gravitational field is required to be applied, to move an object against the earth's gravitational field. In such a case, work is said to be done.

Thus, work is said to be done when the test charge is moved against the electric field.

Consider a positive charge $Q_1$ and its electric field $\bar{E}$. If a positive test charge $Q_t$ is placed in this field, it will move due to the force of repulsion. Let the movement of the charge $Q_t$ is $dl$. The direction in which the movement has taken place is denoted by unit vector $\bar{a}_L$, in the direction of $dl$. This is shown in the Fig. 4.1.




According to Coulomb's law the force exerted by the field $\bar{E}$ is given by,

$$\bar{F} = Q_t \bar{E} \quad \text{N}$$

But the component of this force exerted by the field in the direction of $dl$, is responsible to move the charge $Q_t$, through the distance $dl$.

We know that the component of a vector in the direction of the unit vector is the dot product of the vector with that unit vector. Thus the component of $\bar{F}$ in the direction of unit vector $\bar{a}_L$ is given by,

$$\bar{F}_l = \bar{F} \cdot \bar{a}_L = Q_t \, \bar{E} \cdot \bar{a}_L \quad \text{N}$$

This is the force responsible to move the charge $Q_t$ through the distance $dl$, in the direction of the field.

To keep the charge in equilibrium, it is necessary to apply the force which is equal and opposite to the force exerted by the field in the direction $dl$.

$$\therefore \qquad \bar{F}_{applied} = -\bar{F}_l = Q_t \, \bar{E} \cdot \bar{a}_L \quad \text{N}$$

In this case, the work is said to be done.

**Key Point:** *Thus keeping the charge in equilibrium means we are moving a charge $Q_t$, through the distance $dl$ in opposite direction to that of field $\bar{E}$. Hence the work is done.*

Thus there is expenditure of energy which is given by the product of force and the distance.

Hence mathematically the differential work done by an external source in moving the charge $Q_t$ through a distance $dl$, against the direction of field $\bar{E}$ is given by,

$$dW = \bar{F}_{applied} \times dl = -Q_t \bar{E} \cdot \bar{a}_L \, dl$$

But $$dl \, \bar{a}_L = d\bar{L} = \text{Distance vector}$$

$\therefore$ $$dW = -Q_t \bar{E} \cdot d\bar{L} \ \ J$$

**Key Point:** *Note that dW is a scalar quantity as* $\bar{E} \cdot d\bar{L}$ *is the dot product which is a scalar quantity.*

Thus if a charge Q is moved from initial position to the final position, against the direction of electric field $\bar{E}$ then the total work done is obtained by integrating the differential work done over the distance from initial position to the final position.

$$\therefore \quad W = \int_{\text{Initial}}^{\text{Final}} dW = \int_{\text{Initial}}^{\text{Final}} -Q\,\bar{E} \cdot d\bar{L}$$

$$\therefore \quad \boxed{W = -Q \int_{\text{Initial}}^{\text{Final}} \bar{E} \cdot d\bar{L} \quad J}$$

The work done is measured in joules.

**Key Point:** *Note that at both the positions initial and final, the charge Q is at rest and not moving, then only the equation (7) is valid.*

## The Line Integral

Consider that the charge is moved from initial position B to the final position A, against the electric field $\overline{E}$ then the work done is given by,

$$W = -Q\int_{B}^{A} \overline{E} \cdot d\overline{L}$$

This is called the line integral, where $\overline{E} \cdot d\overline{L}$ gives the component of $\overline{E}$ along the direction $d\overline{L}$.

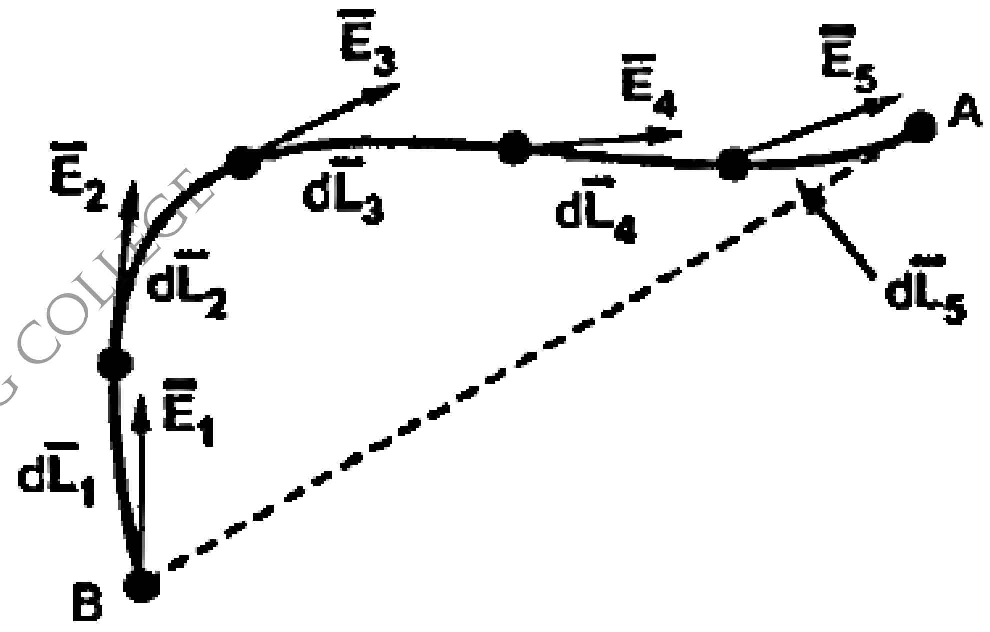
Mathematical procedure involved in such a line integral, is,

1. Choose any arbitrary path B to A.
2. Break up the path into number of very small segments, which are called differential lengths.
3. Find the component of $\overline{E}$ along each segments.
4. Adding all such components and multiplying by charge, the required work done can be obtained.

Thus line integral is basically a summation and accurate result is obtained when the number of segments becomes infinite.

Let us see an important property of this line integral. Consider an uniform electric field $\bar{E}$. The charge is moved from B to A along the path shown in the Fig. 4.2.

The path B to A is divided into number of small segments.




The various distance vectors along the segments choosen are $d\bar{L}_1$, $d\bar{L}_2$, $d\bar{L}_3$, $d\bar{L}_4$ and $d\bar{L}_5$ while the electric field in these directions is $\bar{E}_1$, $\bar{E}_2$, $\bar{E}_3$, $\bar{E}_4$ and $\bar{E}_5$. Hence the line integral from B to A can be expressed as the summation of dot products.

$$\therefore \qquad W = -Q[\bar{E}_1 \cdot d\bar{L}_1 + \bar{E}_2 \cdot d\bar{L}_2 + \ldots + \bar{E}_5 \cdot d\bar{L}_5]$$

But the electric field is uniform and is equal in all directions.

$$\therefore \quad \bar{E}_1 = \bar{E}_2 = \bar{E}_3 = \bar{E}_4 = \bar{E}_5 = \bar{E}$$

$$\therefore \quad W = -Q\bar{E} \cdot [d\bar{L}_1 + d\bar{L}_2 + \ldots + d\bar{L}_5]$$

Now $d\bar{L}_1 + dL_2 + \ldots + d\bar{L}_5$ is the vector addition. So according to method of polygon the sum of all such vectors is the vector joining initial point to final point when all vectors are arranged one after the other in respective directions. This is shown in the Fig. 4.3. Hence the sum of all such vectors is the vector $\bar{L}_{BA}$ joining initial point to final point.

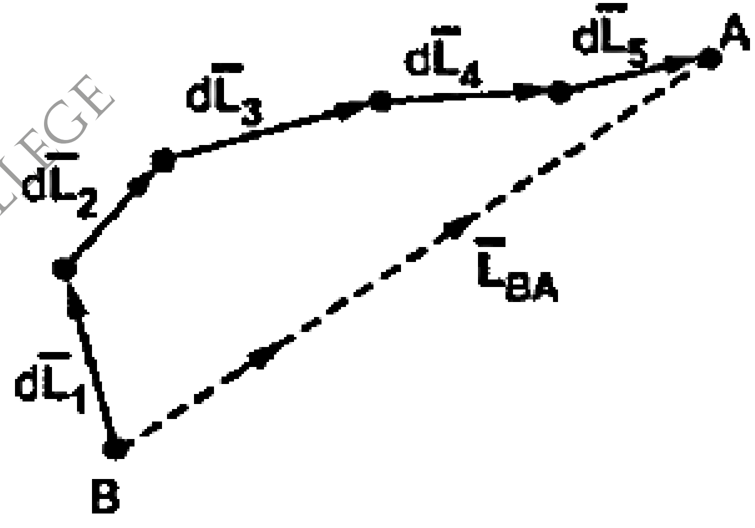



Fig. 4.3

$$\therefore \quad W = -QE \cdot \bar{L}_{BA} \qquad \ldots \text{(Uniform } \bar{E}\text{)}$$

Thus it can be seen that vector sum of small segments choosen along any path, a curve or a straight line remains same as $\bar{L}_{BA}$ and it depends on the initial and final point only.

**Key Point:** *Hence the work done depends on Q, $\bar{E}$ and $\bar{L}_{BA}$ and does not depend on the path joining B to A. This is true for nonuniform electric field $\bar{E}$ as well.*

Thus, the work done in moving a charge from one location B to another A, in a static, uniform or nonuniform electric field $\bar{E}$ is independent of the path selected. The line integral of $\bar{E}$ is determined completely by the endpoints B and A of the path and not the actual path selected.

**Key Point:** *This is called conservative property of electric field* $\bar{E}$ *and field* $\bar{E}$ *is said to be conservative.*

While solving the problems, it is necessary to select $d\bar{L}$ according to the conditions and co-ordinate system selected. The expressions for $d\bar{L}$ in three co-ordinate systems are given here again for the convenience of the readers.

The work done in moving a point charge in an electric field $\bar{E}$ from position B to A is given by,

$$Q = -Q\int_B^A \bar{E} \cdot d\bar{L}$$

1. When the movement of the charge Q is against the direction of E, then the work done is positive, which indicates external source has done the work.

2. When the movement of the charge Q is in the direction of $\bar{E}$, then the work done is negative, which indicates field itself has done the work, no external source is required.

3. The work done is independent of the path selected from B to A but it depends on end points B and A.

4. When the path selected is such that it is always perpendicular to $\bar{E}$ i.e. the force on the charge is always exerted at right angles to the direction in which charge is moving, then the work done is zero. This indicates $\theta$, the angle between $\bar{E}$ and $d\bar{L}$ is 90°. Due to the dot product, the line integral is zero when $\theta = 90°$.

5. If the path selected is such that it is forming a closed contour i.e. starting point is same as the terminating point then the work done is zero.

*An electrostatic field is given by,* $\bar{E} = -8xy\,\bar{a}_x - 4x^2\,\bar{a}_y + \bar{a}_z$ *V/m*

*The charge of 6 C is to be moved from B (1, 8, 5) to A (2, 18, 6). Find the work done in each of the following cases.*

1. *The path selected is* $y = 3x^2 + z,\ z = x + 4$
2. *The straight line from B to A.*

*Show that work done remains same and is independent of the path selected.*

The work done is given by,

$$W = -Q\int_B^A \bar{E}\cdot d\bar{L}$$

Let us differential length $d\bar{L}$ in cartesian co-ordinate system is,

$$d\bar{L} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$$

$$\therefore \quad \bar{E}\cdot d\bar{L} = (-8xy\,\bar{a}_x - 4x^2\,\bar{a}_y + \bar{a}_z)\cdot(dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z)$$

$$= -8xy\,dx - 4x^2\,dy + dz$$

As $\bar{a}_x \cdot \bar{a}_x = \bar{a}_y \cdot \bar{a}_y = \bar{a}_z \cdot \bar{a}_z = 1$, other dot products are zero.

$$W = -Q\int_B^A -8xy\,dx - 4x^2dy + dz$$

$$= -Q\left[\int_B^A -8xy\,dx - \int_B^A 4x^2dy + \int_B^A dz\right]$$

Case 1 : The path is $y = 3x^2 + z, \ z = x + 4$

$$\therefore \qquad y = 3x^2 + x + 4 \qquad \text{differentiate}$$

$$\therefore \qquad dy = (6x+1)\,dx$$

For $\int_B^A -8\,xy\,dx \rightarrow$ The limits are $x = 1$ to $x = 2$.

For $\int_B^A -4x^2 y \rightarrow$ The limits are $y = 8$ to $y = 18$

For $\int_B^A dz \rightarrow$ The limits are $z = 5$ to $z = 6$.

$$\therefore \qquad W = -Q\left[\int_{x=1}^{2} -8\,xy\,dx - \int_{y=8}^{18} 4\,x^2 dy + \int_{z=5}^{6} dz\right]$$

Using $y = 3x^2 + x + 4$ and $dy = (6x + 1)\,dx$ and changing limits of y from 8 to 18 interms of x from 1 to 2 we get

$$\therefore \qquad W = -Q\left[\int_{x=1}^{2} -8x[3x^2 + x + 4]\,dx - \int_{x=1}^{2} 4\,x^2[6x+1]\,dx + \int_{z=5}^{6} dz\right]$$

$$= -Q\left[\int_{x=1}^{2} [-24x^3 - 8x^2 - 32x]\,dx - \int_{x=1}^{2} (24\,x^3 + 4x^2)\,dx + \int_{z=5}^{6} dz\right]$$

$$= -Q\left[\left(-6x^4 - \frac{8}{3}x^3 - 16x^2 - 6x^4 - \frac{4}{3}x^3\right)_{x=1}^{2} + (z)_5^6\right]$$

$$= -Q\{-256 + 1\} = -6 \times -255 = 1530\,\text{J}$$

**Case 2 :** Straight line path from B to A.

To obtain the equations of the straight line, any two of the following three equations of planes passing through the line are sufficient,

$$B\ (1, 8, 5) \quad \text{and} \quad A\ (2, 18, 6)$$

$$(y - y_B) = \frac{y_A - y_B}{x_A - x_B}(x - x_B)$$

$$(z - z_B) = \frac{z_A - z_B}{y_A - y_B}(y - y_B)$$

$$(x - x_B) = \frac{x_A - x_B}{z_A - z_B}(z - z_B)$$

Using the co-ordinates of A and B,

$$y - 8 = \frac{18-8}{2-1}(x-1)$$

$$\therefore \quad y - 8 = 10\,(x - 1)$$

$$\therefore \quad y = 10x - 2$$

$$\therefore \quad dy = 10\,dx$$

And
$$z - 5 = \frac{6-5}{18-8}(y-8)$$

$$\therefore \quad z - 5 = \frac{1}{10}(y-8)$$

$$\therefore \quad 10\,z = y + 42$$

$$W = -Q\left[\int_{x=1}^{2} -8\,xy\,dx - \int_{y=8}^{18} 4\,x^2 dy + \int_{z=5}^{6} dz\right]$$

$$= -Q\left[\int_{x=1}^{2} -8\,x(10x-2)\,dx - \int_{x=1}^{2} 4\,x^2(10dx) + \int_{z=5}^{6} dz\right]$$

$$= -Q\left\{\left[\frac{-80}{3}x^3 + \frac{16x^2}{2} - \frac{40x^3}{3}\right]_{x=1}^{2} + [z]_5^6\right\}$$

$$= -Q\{-213.33 + 32 - 106.667 + 26.667 - 8 + 13.33 + 1\}$$

$$= -Q\,[-255] = -6 \times -255 = 1530\ \text{J}$$

This shows that irrespective of path selected, the work done in moving a charge from B to A remains same.

**Example 4.2 :** *Consider an infinite line charge along z-axis. Show that the work done is zero if a point charge Q is moving in a circular path of radius $r_1$, centered at the line charge.*

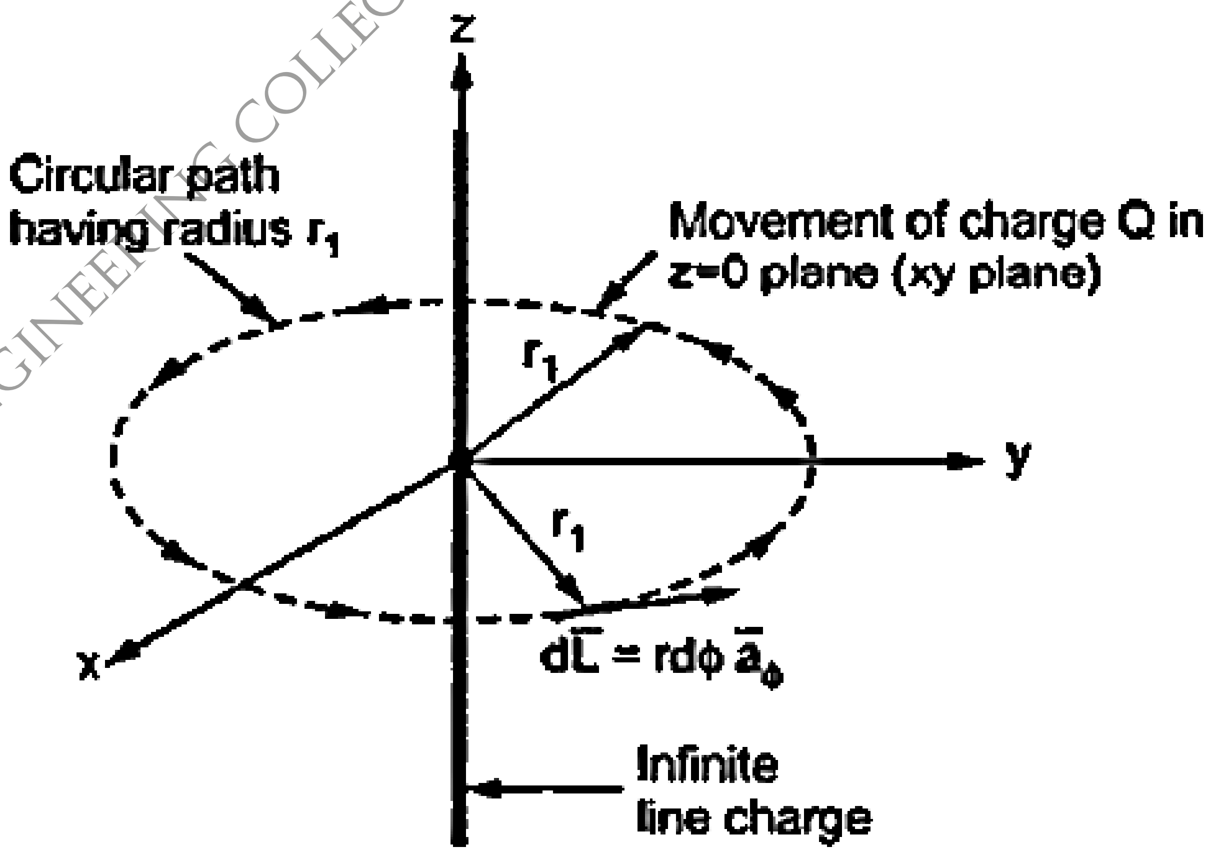
**Solution :** The line charge along the z-axis and the circular path along which charge is moving is shown in the Fig. 4.4.

The circular path is in xy plane such that its radius is $r_1$ and centered at the line charge.

Consider cylindrical co-ordinate system where line charge is along z-axis.

The charge is moving in $\bar{a}_\phi$ direction.

$$\therefore \qquad d\bar{L} = r\,d\phi\,\bar{a}_\phi$$



The field $\bar{E}$ due to infinite line charge along z-axis is given in cylindrical co-ordinates as,

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r$$

The circular path indicates that $d\bar{L}$ has no component in $\bar{a}_r$ and $\bar{a}_z$ direction.

$$d\bar{L} = r\,d\phi\,\bar{a}_\phi$$

$$W = -Q\int_{\text{initial}}^{\text{final}} \bar{E}\bullet d\bar{L} = -Q\int_0^{2\pi} \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r \bullet r\,d\phi\,\bar{a}_\phi$$

$$= -Q\int_0^{2\pi}\frac{\rho_L}{2\pi\varepsilon_0}d\phi\,(\bar{a}_r \bullet \bar{a}_\phi) = 0$$

As $\bar{a}_r \bullet \bar{a}_\phi = 0$ as $\theta = 90°$ between $\bar{a}_r$ and $\bar{a}_\phi$.

This shows that the work done is zero while moving a charge such that path is always perpendicular to the $\bar{E}$ direction.

**Potential Difference**

In the last sections it has been discussed that the work done in moving a point charge Q from point B to A in the electric field $\bar{E}$ is given by,

$$W = -Q\int_{B}^{A} \bar{E} \cdot d\bar{L}$$

If the charge Q is selected as unit test charge then from the above equation we get the work done in moving unit charge from B to A in the field $\bar{E}$. This work done in moving unit charge from point B to A in the field $\bar{E}$ is called potential difference between the points B and A. It is denoted by V.

$$\therefore \text{ Potential difference} = V = -\int_{B}^{A} \bar{E} \cdot d\bar{L}$$

**Thus work done per unit charge in moving unit charge from B to A in the field $\bar{E}$ is called potential difference between the points B and A.**

**Notation :** If B is the initial point and A is the final point then the potential difference is denoted as $V_{AB}$ which indicates the potential difference between the points A and B and unit charge is moved from B to A.

$$\therefore \qquad \boxed{V_{AB} = -\int_{B}^{A} \bar{E} \cdot d\bar{L}}$$

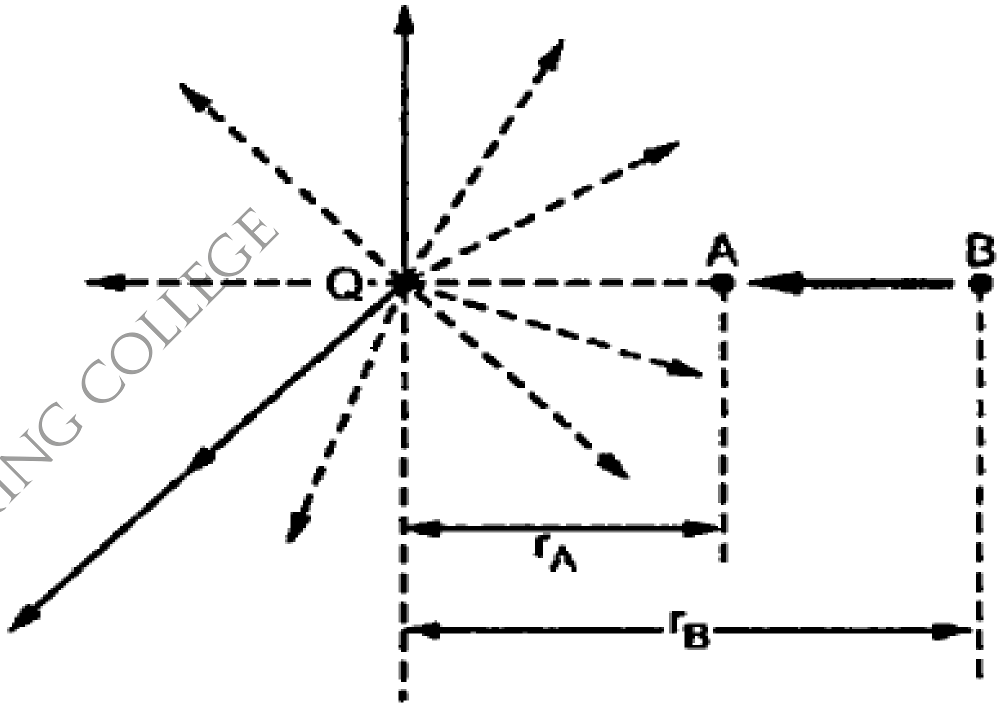
**Key Point:** $V_{AB}$ *is positive if the work is done by the external source in moving the unit charge from B to A, against the direction of* $\bar{E}$.

One volt potential difference is one joule of work done in moving unit charge from one point to other in the field $\bar{E}$.

$$\therefore \qquad 1 \text{ volt} = \frac{1 \text{ joule}}{1 \text{ coulomb}}$$

## Potential due to Point Charge

Consider a point charge, located at the origin of a spherical co-ordinate system, producing $\bar{E}$ radially in all the directions as shown in the Fig. 4.6.




Assuming free space, the field $\bar{E}$ due to a point charge Q at a point having radial distance r from origin is given by,

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}\bar{a}_r \qquad \text{... (1)}$$

Consider a unit charge which is placed at a point B which is at a radial distance of $r_B$ from the origin. It is moved against the direction of E from point B to point A. The point A is at a radial distance of $r_A$ from the origin.

The differential length in spherical system is,

$$d\bar{L} = dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\,d\phi\,\bar{a}_\phi$$

Hence the potential difference $V_{AB}$ between points A and B is given by,

$$V_{AB} = -\int_{B}^{A} \bar{E}\cdot d\bar{L} \qquad \text{But } B \Rightarrow r_B \quad \text{and} \quad A \Rightarrow r_A$$

$$\therefore \qquad V_{AB} = -\int_{r_B}^{r_A} \left(\frac{Q}{4\pi\varepsilon_0 r^2}\bar{a}_r\right)\cdot(dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\,d\phi\,\bar{a}_\phi)$$

$$\therefore \qquad V_{AB} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2}\,dr$$

$$\therefore \qquad V_{AB} = -\frac{Q}{4\pi\varepsilon_0}\int_{r_B}^{r_A} r^{-2}dr = \frac{-Q}{4\pi\varepsilon_0}\left[\frac{r^{-1}}{-1}\right]_{r_B}^{r_A}$$

$$= -\frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_{r_B}^{r_A} = \frac{-Q}{4\pi\varepsilon_0}\left[-\frac{1}{r_A}-\left(-\frac{1}{r_B}\right)\right]$$

$$\therefore \quad \boxed{V_{AB} = -\frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{r_A}+\frac{1}{r_B}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A}-\frac{1}{r_B}\right] V} \quad \dots (4)$$

When $r_B > r_A$, $\frac{1}{r_B} < \frac{1}{r_A}$ and $V_{AB}$ is positive. This indicates the work is done by external source in moving unit charge from B to A.

## Concept of Absolute Potential

Instead of potential difference, it is more convenient to express absolute potentials of various points in the field. Such absolute potentials are measured with respect to a specified reference position. Such a reference position is assumed to be at zero potential.

Consider potential difference $V_{AB}$ due to movement of unit charge from B to A in a field of a point charge Q. It is given by equation (4).

$$V_{AB} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]$$

Now let the charge is moved from infinity to the point A i.e. $r_B = \infty$. Hence $\frac{1}{r_B} = \frac{1}{\infty} = 0$.

$$\therefore \quad V_{AB} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{\infty}\right] = \frac{Q}{4\pi\varepsilon_0 r_A}\ \text{V} \qquad \text{... (5)}$$

The quantity represented by equation (5) is called **potential of point A** denoted as $V_A$.

$$\therefore \quad \boxed{V_A = \frac{Q}{4\pi\varepsilon_0 r_A}\ \text{V}} \qquad \text{... (6)}$$

This is also called **absolute potential** of point A.

Similarly absolute potential of point B can be defined as,

$$\therefore \quad \boxed{V_B = \frac{Q}{4\pi\varepsilon_0 r_B}\ \text{V}} \qquad \text{... (7)}$$

This is work done in moving unit charge from infinity at point B.

Hence the potential difference can be expressed as the difference between the absolute potentials of the two points.

$$\therefore \qquad \boxed{V_{AB} = V_A - V_B \quad V} \qquad \text{... (8)}$$

Thus absolute potential can be defined as,

The absolute potential at any point in an electric field is defined as the work done in moving a unit test charge from the infinity (or reference point at which potential is zero) to the point, against the direction of the field.

Hence absolute potential at any point which is at a distance r from the origin of a spherical system, where point charge Q is located, is given by,

$$\boxed{V = \frac{Q}{4\pi\varepsilon_0 r}} \qquad \text{... (9)}$$

The reference point is at infinity.

**Key Point:** *Note that the potential is a scalar quantity.*

## Potential due to Point Charge not at Origin

If the point charge Q is not located at the origin of a spherical system then obtain the position vector $r'$ of the point where Q is located.

Then the absolute potential at a point A located at a distance r from the origin is given by,

$$V(r) = V_A = \frac{Q}{4\pi\varepsilon_0 |r - r'|}$$

$$= \frac{Q}{4\pi\varepsilon_0 R_A} \quad \ldots (12)$$

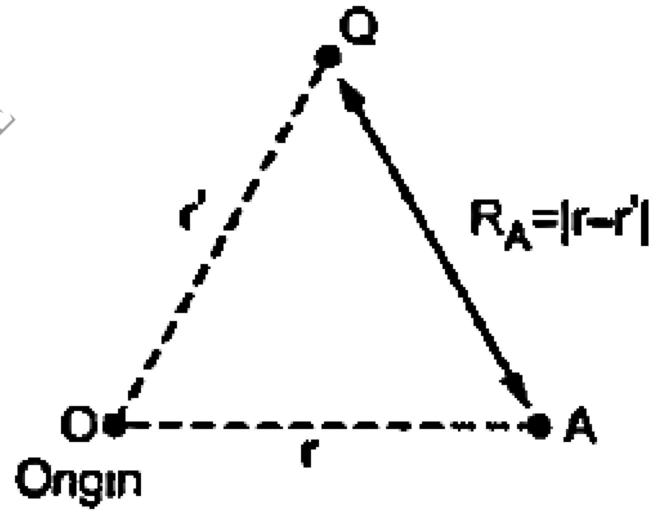



Fig. 4.8

where $R_A = |r - r'|$ = Distance between point at which potential is to be calculated and the location of the charge

**Key Point:** *R is only the distance and not the vector. The potential is a scalar quantity hence only distance* $R = |r - r'|$ *is involved in the determination of potential of point A. The reference is still infinity.*

**Potential due to Several Point Charges**

Consider the various point charges $Q_1, Q_2 \dots Q_n$ located at the distances $r_1, r_2 \dots r_n$ from the origin as shown in the Fig. 4.9. The potential due to all these point charges, at point A is to be determined. Use superposition principle.

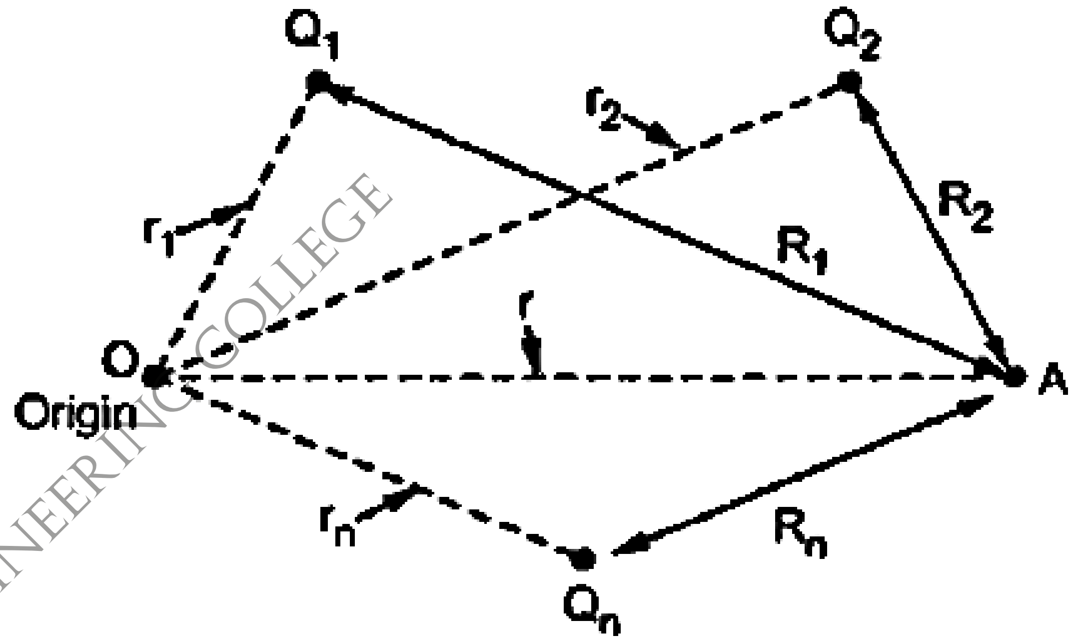



**Fig. 4.9 Potential due to several point charges**

Consider the point charge $Q_1$.

The potential $V_{A1}$ due to $Q_1$ is given by,

$$V_{A1} = \frac{Q}{4\pi\varepsilon_0 |r - r_1|} = \frac{Q}{4\pi\varepsilon_0 R_1} \text{ V}$$

where $R_1 = |r - r_1|$ = Distance between point A and position of $Q_1$

The potential $V_{A2}$ due to $Q_2$ is given by,

$$V_{A2} = \frac{Q_2}{4\pi\varepsilon_0 |r - r_2|} = \frac{Q_2}{4\pi\varepsilon_0 R_2} \text{ V}$$

Thus potential $V_{An}$ due to $Q_n$ is given by,

$$V_{An} = \frac{Q_n}{4\pi\varepsilon_0 |r - r_n|} = \frac{Q_n}{4\pi\varepsilon_0 R_n} \text{ V}$$

As the potential is scalar, the net potential at point A is the algebraic sum of the potentials at A due to individual point charges, considered one at a time.

$$\therefore \quad V(r) = V_A = V_{A1} + V_{A2} + \ldots + V_n$$

$$= \frac{Q_1}{4\pi\varepsilon_0 R_1} + \frac{Q_2}{4\pi\varepsilon_0 R_2} + \ldots + \frac{Q_n}{4\pi\varepsilon_0 R_n}$$

$$\therefore \quad \boxed{V_A = V(r) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\varepsilon_0 |r - r_m|} = \sum_{m=1}^{n} \frac{Q_m}{4\pi\varepsilon_0 R_m} \text{ V}}$$

**Key Point:** *Note that the reference point of zero potential is still assumed to be at infinity.*

**Potential Calculation When Reference is other than Infinity**

The expressions derived uptill now are under the assumption that the reference position of zero potential is at infinity.

If any other point than infinity is selected as the reference then the potential at a point A due to point charge Q at the origin becomes,

$$\boxed{V_A = \frac{Q}{4\pi\varepsilon_0 R_A} + C}$$

where C = Constant to be determined at choosen reference point where V = 0.

Note that the potential difference between the two points is not the function of C.

**Key Point:** ***Another important note is that if the potential difference is to be calculated then reference is not needed. The reference is important only when the absolute potential is to be calculated.***

**Example 4.8 :** *A point charge of 6 nC is located at origin in free space, find potential of point P if P is located at (0.2, – 0.4, 0.4) and*

*a) V = 0 at infinity*

*b) V = 0 at (1, 0, 0)*

*c) V = 20 V at (– 0.5, 1, – 1).*



**Solution :** a) The reference is at infinity, hence

$$V_P = \frac{Q}{4\pi\varepsilon_0 R_P}$$

$$R_P = \sqrt{(0.2-0)^2+(-0.4-0)^2+(0.4-0)^2}$$

$$= 0.6$$

$$\therefore V_P = \frac{6\times10^{-9}}{4\pi\times8.854\times10^{-12}\times0.6} = 89.8774\ \text{V}$$

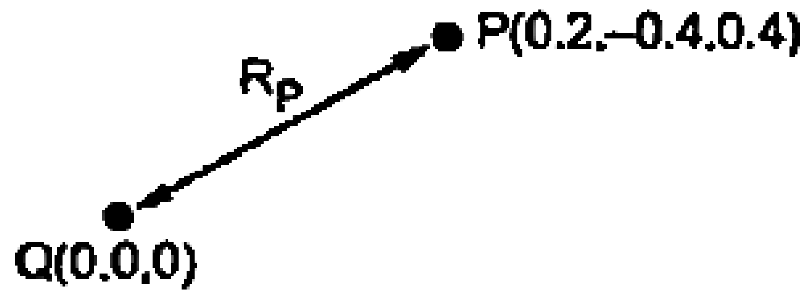



Fig. 4.12

b) V = 0 at (1, 0, 0). Thus the reference is not at infinity. In such a case potential at P is,

$$V_P = \frac{Q}{4\pi\varepsilon_0 R_P} + C$$

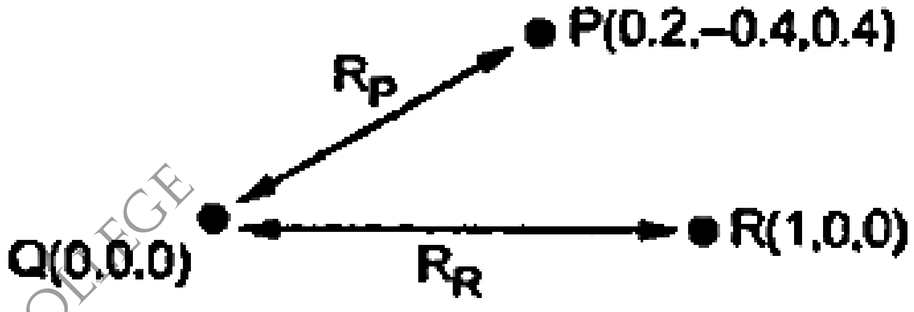



Fig. 4.13

Now $V_R$ at (1, 0, 0) is zero.

$$\therefore \quad V_R = \frac{Q}{4\pi\varepsilon_0 R_R} + C = 0$$

and $$R_R = \sqrt{(1-0)^2 + (0)^2 + (0)^2} = 1$$

$$\therefore \quad 0 = \frac{6\times10^{-9}}{4\pi\times8.854\times10^{-12}\times1} + C$$

$$\therefore \quad C = -53.9264$$

$$\therefore \quad V_P = \frac{Q}{4\pi\varepsilon_0 R_P} + C = 89.8774 - 53.9264 = 35.9509 \text{ V}$$

This is with reference to (1, 0, 0) where V = 0 V.

c) Now V = 20 V at (– 0.5, 1, –1). Let this point is M (– 0.5, 1, – 1). The reference is not given as infinity.

$$V_M = \frac{Q}{4\pi\varepsilon_0 R_M} + C$$

and $V_M = 20 \text{ V}$

while $R_M = \sqrt{(-0.5)^2 + (1)^2 + (-1)^2} = 1.5$

$$\therefore \quad 20 = \frac{6 \times 10^{-9}}{4\pi \times 8.854 \times 10^{-12} \times 1.5} + C$$

$$\therefore \quad C = -15.9509$$

$$\therefore \quad V_P = \frac{Q}{4\pi\varepsilon_0 R_P} + C = 89.8774 - 15.9509$$

$$\therefore \quad V_P = \mathbf{73.9264 \ V}$$

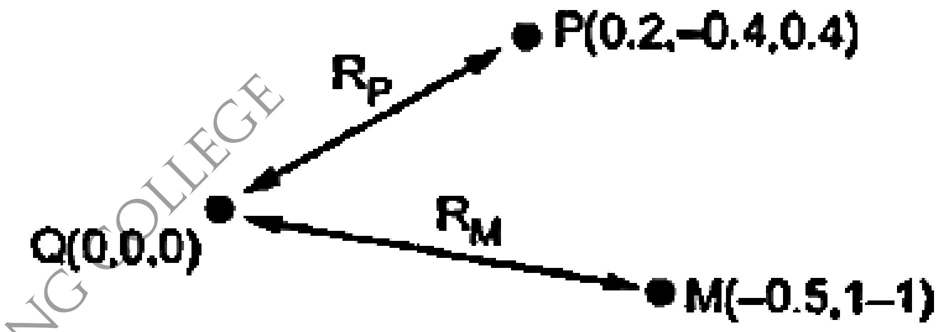



**Fig. 4.14**

**Key Point:** *Note that distance of P from origin where Q is located is $R_P$ which is same in all the cases. Only 'C' changes as the reference changes hence $V_P$ changes.*

**Potential due to a Line Charge**

Consider a line charge having density $\rho_L$ C/m, as shown in the Fig. 4.15.

Consider differential length $dL'$ at a distance $r'$. Then the differential charge on the length $dL'$ is given by,

$$dQ = \rho_L(r')\,dL' \qquad \ldots (1)$$

where $\rho_L(r')$ = Line charge density at $r'$

Let the potential at A is to be determined. Then,

$$dV_A = \frac{dQ}{4\pi\varepsilon_0 |r-r'|} = \frac{dQ}{4\pi\varepsilon_0 R} \qquad \ldots (2)$$

The $R = |r-r'|$ indicates the distance of point A from the differential charge.

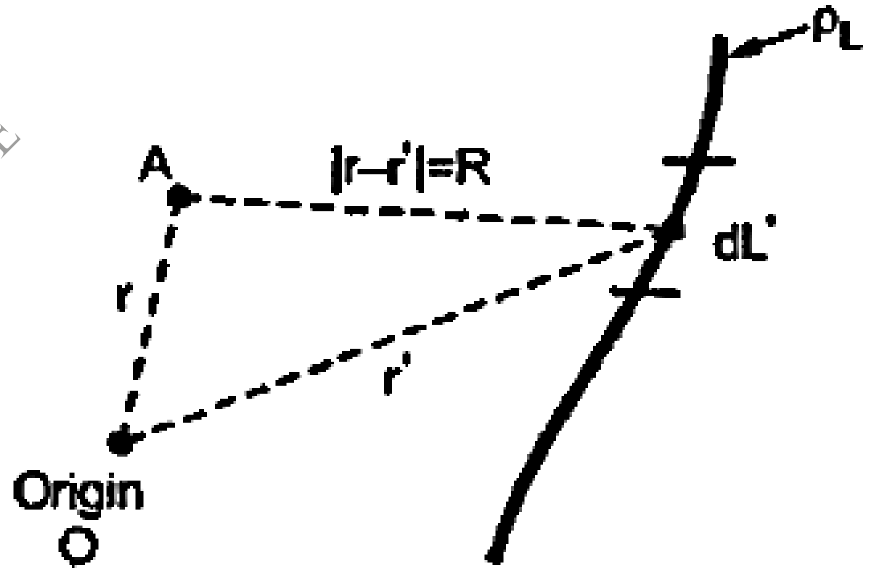



Fig. 4.15

The $dV_A$ is a differential potential at A. Hence the potential $V_A$ can be obtained by integrating $dV_A$ over the length over which line charge is distributed.

$$\therefore \qquad V_A = V(r) = \int_{Line} \frac{dQ}{4\pi\varepsilon_0 R} \qquad \text{and using (1),}$$

$$\therefore \qquad \boxed{V_A = V(r) = \int_{Line} \frac{\rho_L(r')\, dL'}{4\pi\varepsilon_0 R} \ \mathrm{V}} \qquad \text{... (3)}$$

**Key Point:** *Note that R is the distance and not the vector and for uniform line charge density* $\rho_L(r') = \rho_L$.

## Potential due to Surface Charge

Consider uniform surface charge density $\rho_S$ C/m$^2$ on a surface, as shown in the Fig. 4.18.

Consider the differential surface area dS′ at point P where $\rho_S$ is indicated as $\rho_S(r')$

The differential charge can be expressed as,

$$dQ = \rho_S(r')\, dS' \quad \text{... (1)}$$

$$\therefore \quad dV_A = \frac{dQ}{4\pi\varepsilon_0 R} = \frac{\rho_S(r')\, dS'}{4\pi\varepsilon_0 R} \quad \text{...}$$

where R = Distance of point A from the differential charge

The total potential at A can be obtained by integrating $dV_A$ over the given surface.

$$\therefore \quad \boxed{V_A = \int_S \frac{\rho_S(r')\, dS'}{4\pi\varepsilon_0 R}\ \text{V}} \quad \text{...}$$

Note that for uniform surface charge density $\rho_S(r') = \rho_S$.

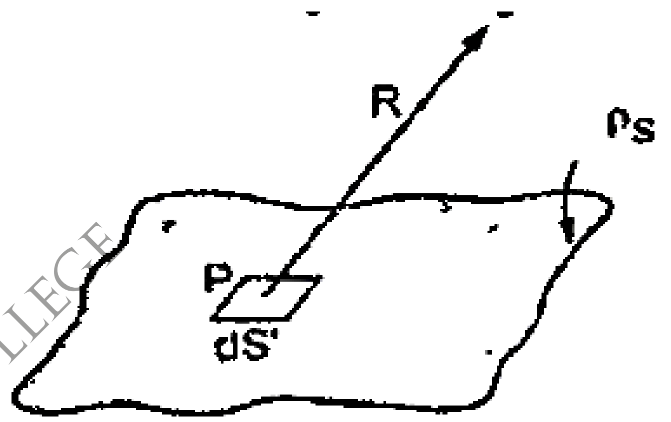



Fig. 4.18 Potential due to surface charge

## Potential due to Volume Charge

Consider a uniform volume charge density $\rho_v$ $C/m^3$ over the given volume as shown in the Fig. 4.19.

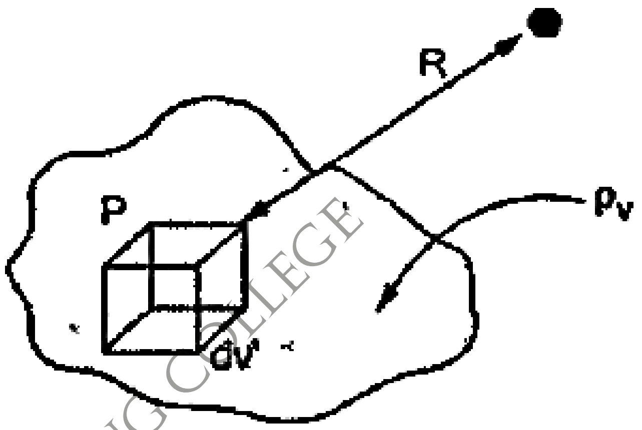



Fig. 4.19 Potential due to volume charge

Consider the differential volume $dv'$ at point P where the charge density is $\rho_v(r')$.

The differential charge can be expressed as,

$$dQ = \rho_v(r')\,dv' \quad \dots (1)$$

$$\therefore \quad dV_A = \frac{dQ}{4\pi\varepsilon_0 R} = \frac{\rho_v(r')\,dv'}{4\pi\varepsilon_0 R}$$

where $R$ = Distance of point A from the differential charge

The total potential at A can be obtained by integrating $dV_A$ over the given volume.

$$\therefore \quad \boxed{V_A = \int_v \frac{\rho_v(r')\,dv'}{4\pi\varepsilon_0 R}\ V}$$

Note that for uniform volume charge density $\rho_v(r') = \rho_v$.

## Potential Difference due to Infinite Line Charge

Consider an infinite line charge along z-axis having uniform line charge density $\rho_L$ C/m.

The point B is at a radial distance $r_B$ while point A is at a radial distance $r_A$ from the charge, as shown in the Fig. 4.22.

The $\bar{E}$ due to infinite line charge along z-axis is known and given by,

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r$$

while $\quad d\bar{L} = dr\,\bar{a}_r$

in cylindrical system in radial direction.

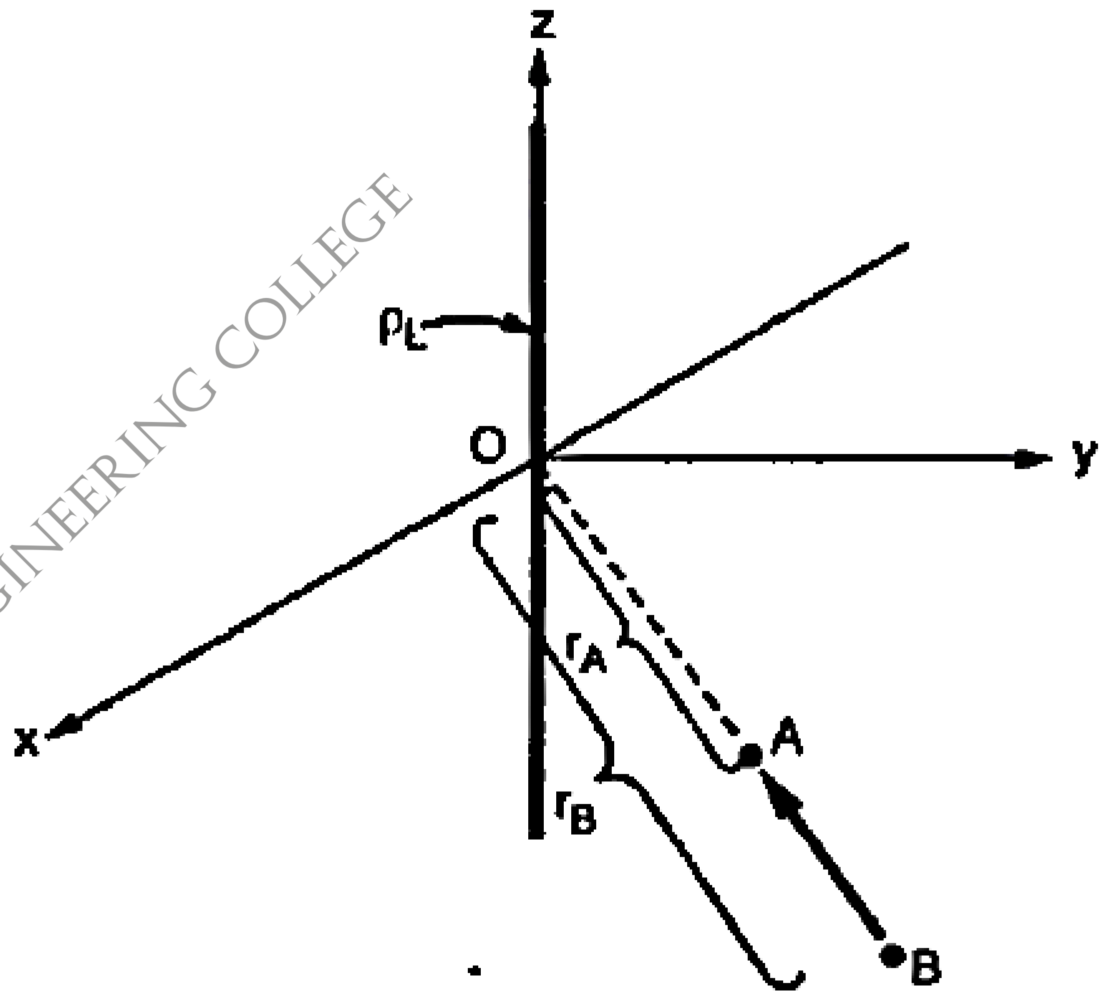



Fig. 4.22

$$V_{AB} = -\int_{B}^{A} \bar{E} \cdot d\bar{L} = -\int_{r_B}^{r_A} \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r \cdot dr\,\bar{a}_r$$

$$= -\int_{r_B}^{r_A} \frac{\rho_L}{2\pi\varepsilon_0 r} dr$$

$$= \frac{-\rho_L}{2\pi\varepsilon_0} \int_{r_B}^{r_A} \frac{1}{r} dr$$

$$= \frac{-\rho_L}{2\pi\varepsilon_0} [\ln r]_{r_B}^{r_A} = \frac{-\rho_L}{2\pi\varepsilon_0} [\ln r_A - \ln r_B]$$

$$\boxed{V_{AB} = \frac{\rho_L}{2\pi\varepsilon_0} \ln\frac{r_B}{r_A}\ \mathrm{V}}$$

## Equipotential Surfaces

In an electric field, there are many points at which the electric potential is same. This is because, the potential is a scalar quantity which depends on the distance between the point at which potential is to be obtained and the location of the charge. There can be number of points which can be located at the same distance from the charge. All such points are at the same electric potential. If the surface is imagined, joining all such points which are at the same potential, then such a surface is called equipotential surface.

**Key Point:** *An equipotential surface is an imaginary surface in an electric field of a given charge distribution, in which all the points on the surface are at the same electric potential.*

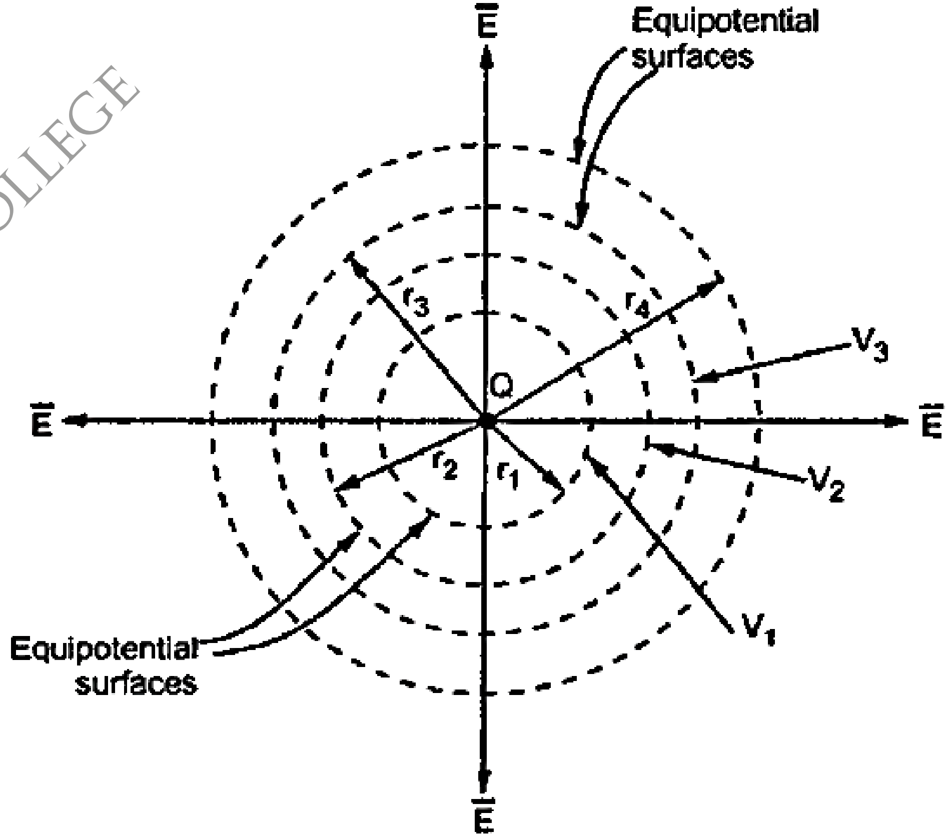
The potential difference between any two points on the equipotential surface is always zero. Thus the work done in moving a test charge from one point to another in an equipotential surface is always zero. There can be many equipotential surfaces existing in an electric field of a particular charge distribution.

Consider a point charge located at the origin of a sphere. Then potential at a point which is at a radial distance r from the point charge is given by,

$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

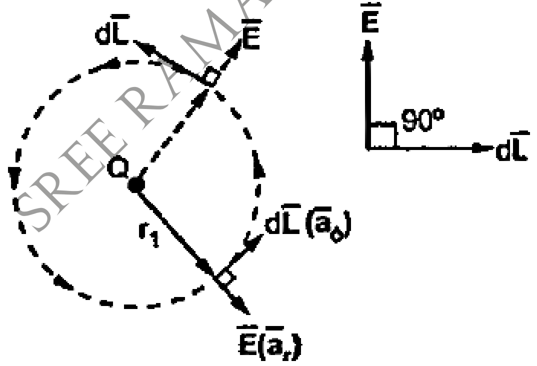
So at all points which are at a distance r from Q, the potential is same and surface joining all such points is equipotential surface.

Similarly at $r = r_1$, $r = r_2$ ... there exists other equipotential surfaces, in an electric field of point charge, in the form of concentric spheres as shown in the Fig. 4.25.

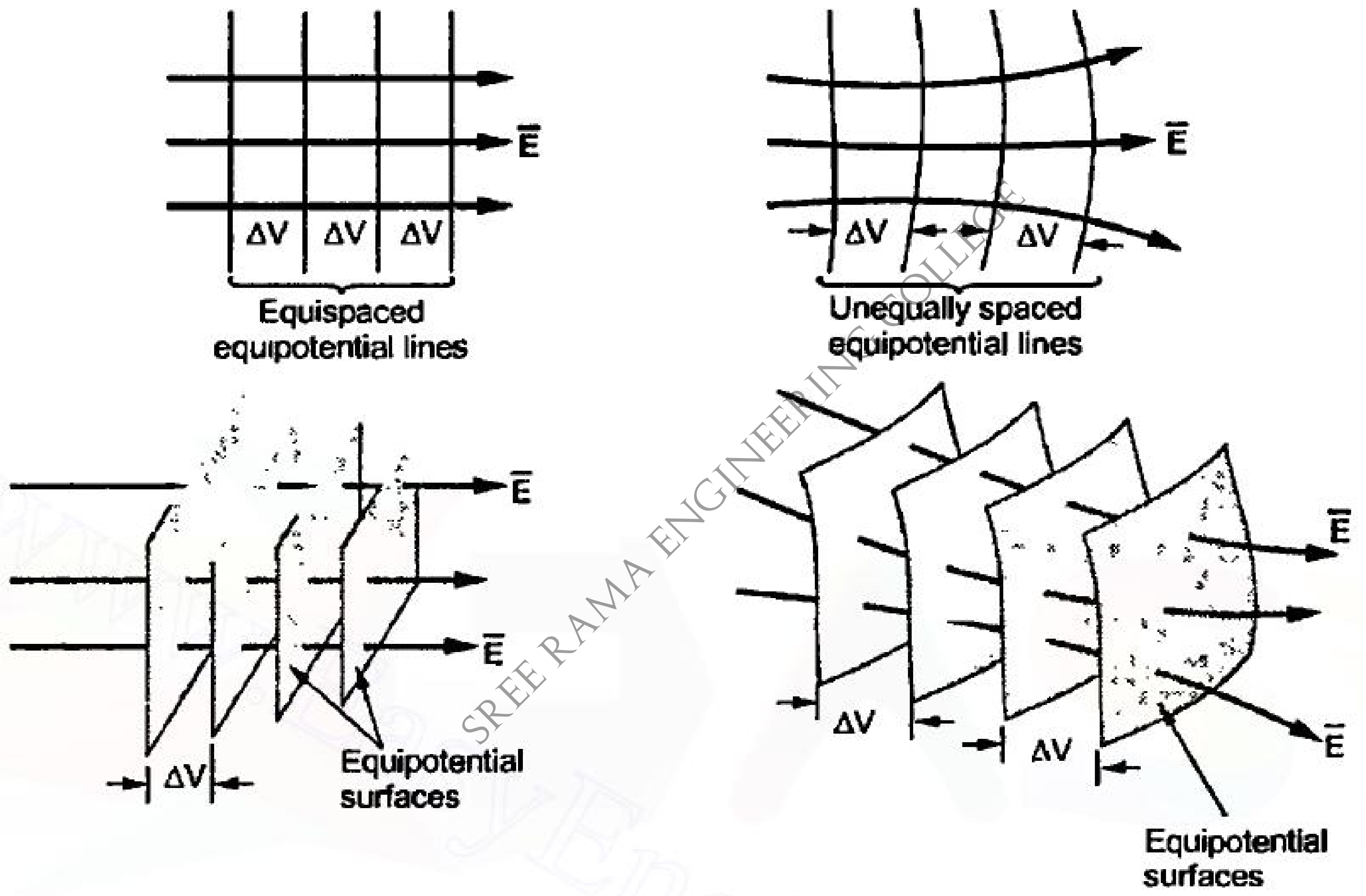



It can be noted that V is inversely proportional to distance r. Thus $V_1$ at equipotential surface at $r = r_1$ is highest and it goes on decreasing, as the distance r increases. Thus $V_1 > V_2 > V_3 > \ldots$.. As we move away from the charge, the $\bar{E}$ decreases hence potential of equipotential surfaces goes on decreasing. While potential of equipotential surfaces goes on increasing as we move against the direction of electric field.

For a uniform field $\bar{E}$, the equipotential surfaces are perpendicular to $\bar{E}$ and are equispaced for fixed increment of voltages. Thus if we move a charge along a circular path of radius $r_1$ as shown in $\bar{a}_\phi$ direction, then work done is zero. This is because $\bar{E}$ and $d\bar{L}$ are perpendicular. Thus $\bar{E}$ and equipotential surface are at right angles to each other.




For a nonuniform field, the field lines tends to diverge in the direction of decreasing $\bar{E}$.




(a) Uniform field

(b) Non-uniform field

## Conservative Field

It is seen that, the work done in moving a test charge around any closed path in a static field $\bar{E}$ is zero. This is because starting and terminating point is same for a closed path. Hence upper and lower limit of integration becomes same hence the work done becomes zero. Such an integral over a closed path is denoted as,

$$\boxed{\oint_{\substack{\text{Closed}\\ \text{path}}} \bar{E} \cdot d\bar{L} = 0} \quad \text{... (1)}$$

**Key Point:** *The $\oint$ sign indicates integral over a closed path. Such a field having property given by equation (1), associated with it, is called* ***conservative field*** *or* ***lamellar field.*** *This indicates that the work done in $\bar{E}$ and hence potential between two points is independent of the path joining the two points.*

**Potential Gradient**

$$V = -\int \bar{E} \cdot d\bar{L} = \frac{Q}{4\pi\varepsilon_0 r}$$

The potential decreases as distance of point from the charge increases. This is shown in the Fig. 4.28.

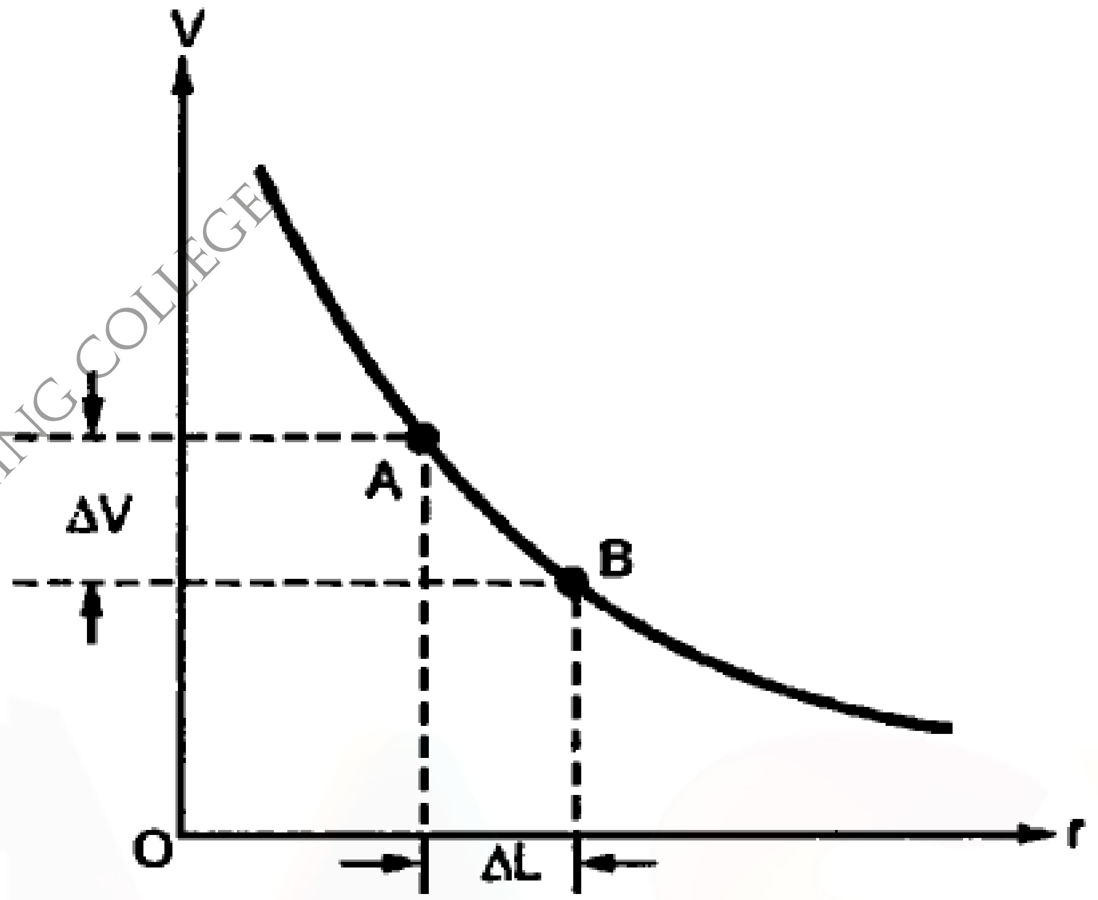
It is known that the line integral of $\bar{E}$ between the two points gives a potential difference between the two points. For an elementary length $\Delta L$ we can write,

$$\therefore \quad V_{AB} = \Delta V = -\bar{E} \cdot \Delta \bar{L}$$

Hence an inverse relation namely the change of potential $\Delta V$, along the elementary length $\Delta L$ must be related to $\bar{E}$, as $\Delta L \to 0$.

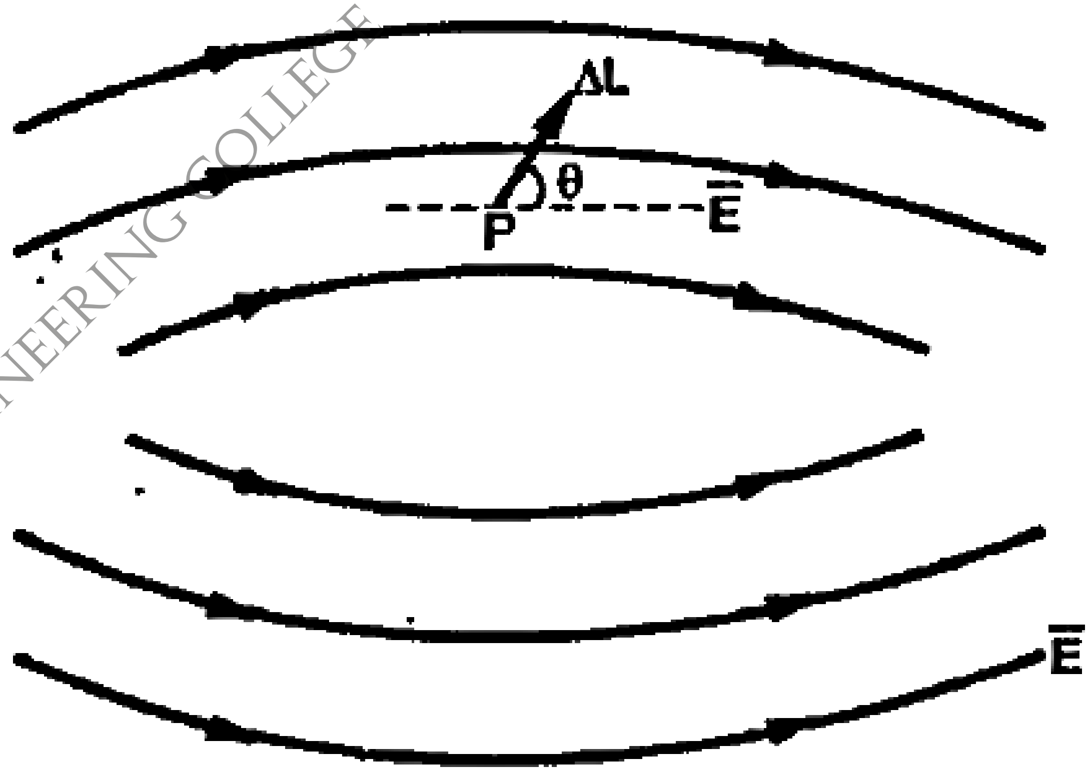
The rate of change of potential with respect to the distance is called the **potential gradient.**

$$\therefore \quad \boxed{\frac{dV}{dL} = \lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L} = \text{Potential gradient}}$$



## Relation between $\bar{E}$ and V

Consider $\bar{E}$ due to a particular charge distribution in space. The electric field $\bar{E}$ and potential V is changing from point to point in space. Consider a vector incremental length $\Delta\bar{L}$ making an angle θ with respect to the direction of $\bar{E}$, as shown in the Fig. 4.29.




To find incremental potential we use,

$$\Delta V = -\bar{E} \cdot \Delta\bar{L} \quad \text{... (1)}$$

Now $$\Delta\bar{L} = \Delta L\, \bar{a}_L \quad \text{... (2)}$$

where $\bar{a}_L$ = Unit vector in the direction of $\Delta L$.

$$\therefore \quad \bar{E} \cdot \Delta \bar{L} = (E_L \bar{a}_L) \cdot (\Delta L \bar{a}_L) \qquad \ldots \bar{a}_L \cdot \bar{a}_L = 1$$

$$\therefore \quad \bar{E} \cdot \Delta \bar{L} = E_L \Delta L$$

$$\therefore \quad \Delta V = -E_L \Delta L \qquad \ldots (3)$$

where $E_L$ = Component of $\bar{E}$ in the direction of $\bar{a}_L$.

In other words, dot product can be expressed in terms of cos θ as.

$$\Delta V = -E \Delta L \cos\theta \qquad \ldots \text{as } \bar{E} \cdot \Delta \bar{L} = |E| |\Delta L| \cos\theta$$

$$\therefore \quad \frac{\Delta V}{\Delta L} = -E \cos\theta \qquad \ldots (4)$$

To find ΔV at a point, take $\lim \Delta L \to 0$.

$$\therefore \quad \lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L} = -E \cos\theta \qquad \ldots (5)$$

$$\text{But} \quad \lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L} = \frac{dV}{dL} = \text{Potential gradient}$$

$$\therefore \qquad \frac{dV}{dL} = -E\cos\theta \qquad \dots (6)$$

At a point P where $\Delta L$ is considered, $\bar{E}$ has a fixed value while $\Delta L$ is also constant. Hence potential gradient $\frac{dV}{dL}$ can be maximum only when $\cos\theta = -1$ i.e. $\theta = +180°$. This indicates that $\Delta L$ must be in the direction opposite to $\bar{E}$.

$$\therefore \qquad \left.\frac{dV}{dL}\right|_{max} = E \qquad \dots (7)$$

This equation shows that,

1. Maximum value of the potential gradient gives the magnitude of the electric field intensity $\bar{E}$.

2. The maximum value of rate of change of potential with distance i.e. potential gradient is possible only when the direction of increment in distance is opposite to the direction of $\bar{E}$.

Thus if $\bar{a}_n$ is the unit vector in the direction of increasing potential normal to the equipotential surface then $\bar{E}$ can be expressed as,

$$\bar{E} = -\left.\frac{dV}{dL}\right|_{max} \bar{a}_n \qquad \dots (8)$$

$$\bar{E} = -\left.\frac{dV}{dL}\right|_{max} \bar{a}_n \qquad \text{... (8)}$$

As $\bar{E}$ and potential gradient are in opposite direction, equation (8) has a negative sign.

The equation shows that the magnitude of $\bar{E}$ is given by maximum space rate of change of V while the direction of $\bar{E}$ is normal to the equipotential surface in the direction of decreasing potential.

**The maximum value of rate of change of potential with distance (dV/dL) is called gradient of V.**

The mathematical operation on V by which $-\bar{E}$ is obtained is called gradient and denoted as,

$$\text{Gradient of V} = \text{grad V} = \nabla V \qquad \text{... (9)}$$

$$\therefore \quad \nabla V = \text{grad V} = -\bar{E} \ \text{V/m} \qquad \text{... (10)}$$

$$\text{or} \quad \boxed{\bar{E} = -\nabla V = -(\text{grad V})} \qquad \text{... (11)}$$

The equation (11) gives the relationship between $\bar{E}$ and V. Now $\bar{E}$ is vector but V is scalar, hence remember that grad V i.e. gradient of a scalar is a vector.

The grad V in various co-ordinate systems are,

| Sr. No. | Co-ordinate system | Grad V = ∇ V |
|---|---|---|
| 1. | Cartesian | $\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}a_z$ |
| 2. | Cylindrical | $\nabla V = \frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\bar{a}_\phi + \frac{\partial V}{\partial z}\bar{a}_z$ |
| 3. | Spherical | $\nabla V = \frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi$ |

*If* $V = 2x^2y + 20z - \dfrac{4}{x^2 + y^2}$ *V* *Find E,* $\vec{D}$ *and* $\rho_v$ *at P (6, − 2.5, 3).*

$$\overline{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\overline{a}_x + \frac{\partial V}{\partial y}\overline{a}_y + \frac{\partial V}{\partial z}\overline{a}_z\right]$$

$$\frac{\partial V}{\partial x} = 2y(2x) + 0 - 4\left[\frac{-(2x)}{(x^2+y^2)^2}\right] = 4xy + \frac{8x}{(x^2+y^2)^2}$$

$$\frac{\partial V}{\partial y} = 2x^2 + 0 - 4\left[\frac{-2y}{(x^2+y^2)^2}\right] = 2x^2 + \frac{8y}{(x^2+y^2)^2}$$

$$\frac{\partial V}{\partial z} = 0 + 20 - 0 = 20$$

$$\overline{E} = -\left\{\left[4xy + \frac{8x}{(x^2+y^2)^2}\right]\overline{a}_x + \left[2x^2 + \frac{8y}{(x^2+y^2)^2}\right]\overline{a}_y + 20\overline{a}_z\right\}$$

$$\overline{E} = -\left\{\left[4xy + \frac{8x}{(x^2+y^2)^2}\right]\overline{a}_x + \left[2x^2 + \frac{8y}{(x^2+y^2)^2}\right]\overline{a}_y + 20\,\overline{a}_z\right\}$$

$$\overline{E} \text{ at } P = -\{[-60 + 0.0268]\overline{a}_x + [72 - 0.0112]\overline{a}_y + 20\,\overline{a}_z\}$$

$$= +59.9732\,\overline{a}_x - 71.9888\,\overline{a}_y - 20\,\overline{a}_z \quad \text{V/m}$$

$$\overline{D} \text{ at } P = \overline{E} \text{ at } P \times \varepsilon_0$$

$$= 0.531\,\overline{a}_x - 0.6373\,\overline{a}_y - 0.177\,\overline{a}_z \quad \text{nC/m}^2$$

$$\rho_v = \nabla \cdot \overline{D}$$

$$\overline{D} = \varepsilon_0 E \qquad \text{hence} \quad \rho_v = (\nabla \cdot \overline{E})\varepsilon_0$$

$$\nabla \cdot \overline{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$= -\frac{\partial}{\partial x}\left[4xy + \frac{8x}{(x^2+y^2)^2}\right] - \frac{\partial}{\partial y}\left[2x^2 + \frac{8y}{(x^2+y^2)^2}\right] - \frac{\partial}{\partial z}(20)$$

$$= -\frac{\partial}{\partial x}\left[4xy+\frac{8x}{(x^2+y^2)^2}\right]-\frac{\partial}{\partial y}\left[2x^2+\frac{8y}{(x^2+y^2)^2}\right]-\frac{\partial}{\partial z}(20)$$

$$= -\left[4y+\left\{\frac{(x^2+y^2)^2\,8-8x\,2\,(x^2+y^2)\,(2x)}{(x^2+y^2)^4}\right\}\right]-\left[0+\left\{\frac{(x^2+y^2)^2\,(8)-8y\,2\,(x^2+y^2)\,(2y)}{(x^2+y^2)^4}\right\}\right]-0$$

$$= -4y-\frac{8}{(x^2+y^2)^2}+\frac{32\,x^2}{(x^2+y^2)^3}-\frac{8}{(x^2+y^2)^2}+\frac{32\,y^2}{(x^2+y^2)^3}$$

At P, $x = 6,\ y = -2.5$ and $z = 3$.

$$\nabla\cdot\bar{E} = 10-4.4816\times10^{-3}+0.01527-4.4816\times10^{-3}+2.651\times10^{-3}$$

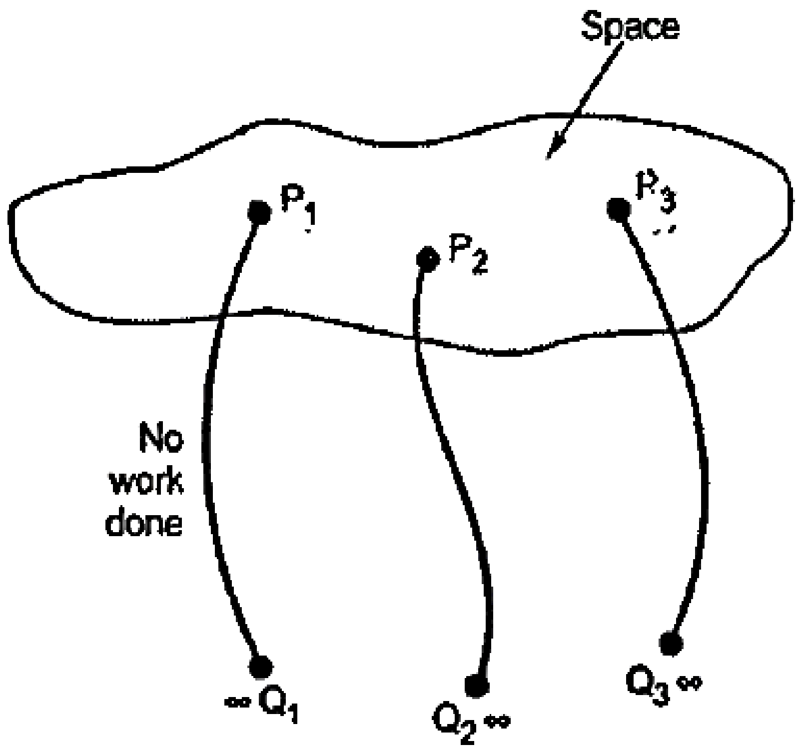
$$= 10.00895$$

$$\rho_v \text{ at } P = \varepsilon_0\,[\nabla\cdot\bar{E}] = 8.854\times10^{\ 12}\times10.00895$$

$$= 88.6193\ \ pC/m^3$$

## Energy Density in the Electrostatic Fields

Consider an empty space where there is no electric field at all. The charge $Q_1$ is moved from infinity to a point in the space say $P_1$. This requires no work as there is no $\bar{E}$ present. Now the charge $Q_2$ is to be placed at point $P_2$ in the space as shown in the Fig. 4.30. But now there is an electric field due to $Q_1$ and $Q_2$ is required to be moved against the field of $Q_1$. Hence the work is required to be done.




Now Potential $=$ Work done per unit charge $\left(\dfrac{W}{Q}\right)$

$\therefore$ Work done $=$ Potential $V \times$ Charge $Q$

$\therefore$ Work done to position $Q_2$ at $P_2 = V_{2,1}\, Q_2$

where $V_{2,1}$ $=$ Potential at $P_2$ due to $P_1$

Now let charge $Q_3$ is to be moved from infinity to $P_3$. There are electric fields due to $Q_1$ and $Q_2$. Hence total work done is due to potential at $P_3$ due to charge at $P_1$ and potential at $P_3$ due to charge at $P_2$.

$$\therefore \quad \text{Work done to position } Q_3 \text{ at } P_3 = V_{3,1}\, Q_3 + V_{3,2}\, Q_3$$

Thus for charge $Q_n$ to be placed at $P_n$, we can write,

$$\therefore \quad \text{Work done to position } Q_n \text{ at } P_n = V_{n,1}\, Q_n + V_{n,2}\, Q_n + \ldots$$

Hence the total work done in positioning all the charges is,

$$W_E = Q_2\, V_{2,1} + Q_3\, V_{3,1} + Q_3\, V_{3,2} + \ldots$$

The total work done is nothing but the potential energy in the system of charges hence denoted as $W_E$.

If charges are placed in reverse order we can write,

$$W_E = Q_3\ V_{3,4} + Q_2\ V_{2,3} + Q_2\ V_{2,4} + Q_1\ V_{1,2} + Q_1\ V_{1,3} + Q_1\ V_{1,4} + \ldots$$

In this expression $Q_n$ is placed first, then $Q_{n-1}$ ... then $Q_4, Q_3, Q_2$ and finally $Q_1$.

$$\begin{aligned} 2\,W_E &= Q_1\left(V_{1,2} + V_{1,3} + V_{1,4} + \ldots + V_{1,n}\right) \\ &\quad + Q_2\left(V_{2,1} + V_{2,3} + V_{2,4} + \ldots + V_{2,n}\right) \\ &\quad + Q_3\left(V_{3,1} + V_{3,2} + V_{3,4} + \ldots + V_{3,n}\right) + \ldots \end{aligned}$$

Each sum of the potentials is the total resultant potential due to all the charges except for the charge at the point at which potential is obtained.

$$\therefore \quad V_{1,2} + V_{1,3} + V_{1,4} + \ldots\ldots + V_{1,n} = V_1$$

This is potential at $P_1$ where $Q_1$ is placed due to all other charges $Q_2, Q_3, \ldots Q_n$.

Similarly, $V_{2,1} + V_{2,3} + V_{2,4} + \ldots + V_{2,n} = V_2$ and so on.

$$2W_E = Q_1 V_1 + Q_2 V_2 + Q_3 V_3 + \ldots$$

$$\boxed{W_E = \frac{1}{2} \sum_{m=1}^{n} Q_m V_m \quad J}$$

This is the potential energy stored in the system of n point charges.

For line charge $\rho_L$, $$W_E = \frac{1}{2} \int \rho_l \, dL \, V \quad J$$

For surface charge $\rho_S$, $$W_E = \frac{1}{2} \int \rho_S \, dS \, V \quad J$$

For volume charge $\rho_v$, $$W_E = \frac{1}{2} \int \rho_v \, dv \, V \quad J$$

## Energy Stored Interms of $\overline{D}$ and $\overline{E}$

Consider the volume charge distribution having uniform charge density $\rho_v\ C/m^3$.

$$W_E = \frac{1}{2}\int_{vol} \rho_v\ V\,dv$$

According to Maxwell's first equation,

$$\rho_v = \nabla \cdot \overline{D}$$

$$W_E = \frac{1}{2}\int_{vol} (\nabla \cdot \overline{D})\ V\,dv$$

For any vector A and scalar V there is vector identity,

$$\nabla \cdot V\overline{A} = \overline{A} \cdot \nabla V + V(\nabla \cdot \overline{A})$$

$$\therefore \quad (\nabla \cdot \overline{A})\ V = \nabla \cdot V\overline{A} - \overline{A} \cdot \nabla V$$

$$W_E = \frac{1}{2}\int_{vol} (\nabla \cdot V\overline{D} - \overline{D} \cdot \nabla V)\,dv$$

$$W_E = \frac{1}{2} \int_{vol} (\nabla \cdot V\overline{D} - \overline{D} \cdot \nabla)\, dv$$

$$W_E = \frac{1}{2} \int_{vol} (\nabla \cdot V\overline{D})\, dv - \frac{1}{2} \int_{vol} \overline{D} \cdot \nabla V \; dv$$

According to divergence theorem, volume integral can be converted to closed surface integral if closed surface totally surrounds the volume.

$$\frac{1}{2} \int_{vol} (\nabla \cdot V\overline{D})\, dv = \frac{1}{2} \oint (V\overline{D}) \cdot d\overline{S}$$

$$W_E = \frac{1}{2} \oint (V\overline{D}) \cdot d\overline{S} - \frac{1}{2} \int_{vol} \overline{D} \cdot \nabla V \; dv$$

We know that $V \propto \frac{1}{r}$ and $\overline{D} \propto \frac{1}{r^2}$ for point charge, $V \propto \frac{1}{r^2}$, $\overline{D} \propto \frac{1}{r^3}$ for dipoles and so on. So $V\overline{D}$ is proportional to at least $1/r^3$ while dS varies as $r^2$. Hence total integral varies as 1/r. As surface becomes very large, $r \to \infty$ and $1/r \to 0$. Hence closed surface integral is zero in the equation (16).

$$W_r = -\frac{1}{2} \int_{vol} \overline{D} \bullet \nabla V \; dv$$

$$\overline{E} = -\nabla V$$

$$W_E = -\frac{1}{2} \int_{vol} \overline{D} \bullet (-\overline{E}) \, dv$$

$$\boxed{W_E = \frac{1}{2} \int_{vol} \overline{D} \bullet \overline{E} \; dv \;\; J}$$

Now $\qquad \bar{D} = \varepsilon_0 \bar{E}$

$$W_E = \frac{1}{2}\int_{vol} \varepsilon_0 \bar{E}\cdot\bar{E}\ dv \quad J$$

$$\boxed{W_E = \frac{1}{2}\int_{vol} \varepsilon_0 E^2\ dv \quad J}$$

$$\boxed{W_E = \frac{1}{2}\int_{vol} \frac{D^2}{\varepsilon_0}\ dv \quad J}$$

In a differential form,

$$d W_E = \frac{1}{2}\ \bar{D}\cdot\bar{E}\ dv$$

$$\therefore \qquad \boxed{\frac{d W_E}{dv} = \frac{1}{2}\ \bar{D}\cdot\bar{E}\ \ J/m^3}$$

This is called energy density in the electric field having units $J/m^3$. If this is integrated over the volume, we get total energy present.

$$\boxed{W_E = \int_{vol}\left(\frac{d W_E}{dv}\right) dv}$$

# Maxwell’s Second Equation

It is seen that, the work done in moving a test charge around any closed path in a static field $\bar{E}$ is zero. This is because starting and terminating point is same for a closed path. Hence upper and lower limit of integration becomes same hence the work done becomes zero. Such an integral over a closed path is denoted as,

$$\oint_{\text{Closed path}} \bar{E} \cdot d\bar{L} = 0$$

$$\oint_{L} \bar{E} \cdot d\bar{L} = \int_{S} (\nabla \times \bar{E}) \cdot d\bar{S}$$

# Stoke’s Theorem:

The line integral of $\bar{F}$ around a closed path L is equal to the integral of curl of $\bar{F}$ over the open surface S enclosed by the closed path L.

Mathematically it is expressed as,

$$\oint_{L} \bar{F} \cdot d\bar{L} = \int_{S} (\nabla \times \bar{F}) \cdot d\bar{S}$$

*If $V = x - y + xy + z$ V, find $\bar{E}$ at (1, 2, 4) and the electrostatic energy stored in a cube of side 2 m centered at the origin.*

$$V = x - y + xy + z$$

$$\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right]$$

$$\frac{\partial V}{\partial x} = 1 + y, \frac{\partial V}{\partial y} = -1 + x, \quad \frac{\partial V}{\partial z} = 1$$

$$\bar{E} = -[(1 + y)\bar{a}_x + (x - 1)\bar{a}_y + \bar{a}_z]$$

At (1, 2, 4); $\bar{E} = -3\bar{a}_x - \bar{a}_z$ V/m

Now $W_E = \frac{1}{2}\int_{vol} \varepsilon_0 |\bar{E}|^2 dv, \ dv = dx\ dy\ dz$

$$|\bar{E}|^2 = (1+y)^2 + (x-1)^2 + 1^2 = 1 + 2y + y^2 + x^2 - 2x + 1 + 1$$

$$= x^2 + y^2 - 2x + 2y + 3$$

$$\bar{E} = -[(1+y)\,\bar{a}_x + (x-1)\,\bar{a}_y + \bar{a}_z]$$

$$W_E = \frac{\varepsilon_0}{2}\int_{vol}(x^2 + y^2 - 2x + 2y + 3)\,dx\,dy\,dz$$

The cube is centered at origin hence all the variables x, y and z vary from – 1 to + 1.

$$\therefore \quad W_E = \frac{\varepsilon_0}{2}\int_{z=-1}^{1}\int_{y=-1}^{1}\int_{x=-1}^{1}(x^2 + y^2 - 2x + 2y + 3)\,dx\,dy\,dz$$

$$= \frac{\varepsilon_0}{2}\int_{z=-1}^{1}\int_{y=-1}^{1}\left[\frac{x^3}{3} + xy^2 - \frac{2x^2}{2} + 2xy + 3x\right]_{x=-1}^{1} dy\,dz$$

$$= \frac{\varepsilon_0}{2}\int_{z=-1}^{1}\int_{y=-1}^{1}\left[\frac{2}{3} + 2y^2 + 4y + 6\right] dy\,dz$$

$$= \frac{\varepsilon_0}{2} \int_{z=-1}^{1} \left[\frac{2}{3}y + \frac{2y^3}{3} + \frac{4y^2}{2} + 6y\right]_{y=-1}^{1} dz$$

$$= \frac{\varepsilon_0}{2} \int_{z=-1}^{1} \left(\frac{4}{3} + \frac{4}{3} + 12\right) dz$$

$$= \frac{\varepsilon_0}{2} \left[\frac{44z}{3}\right]_{-1}^{1}$$

$$= \frac{88\varepsilon_0}{6}$$

$$= 0.12985 \text{ nJ}$$

*What is the potential at the center of a square with a side $a = 2m$ while charges $2\mu C$, $-4\mu C$, $6\mu C$ and $2\mu C$ are located at its four corners ?*

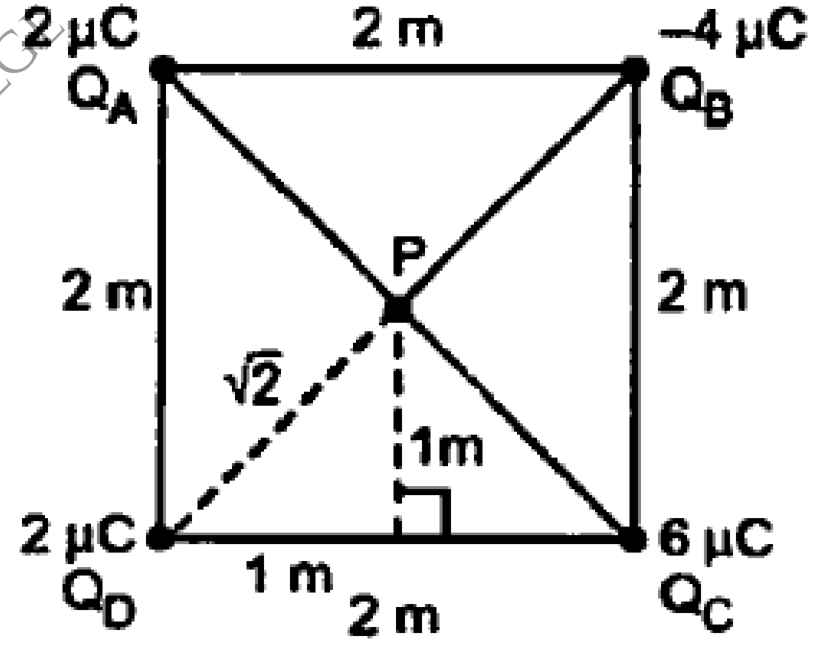
The potential at a point due to a point charge is given by,

$$V = \frac{Q}{4\pi\varepsilon_0 R} \quad \text{where}$$

R = Distance between charge and the point

$$\therefore\ V_{P1} = \text{Potential of P due to } Q_A = \frac{Q_A}{4\pi\varepsilon_0 R_1}$$

where $\quad R_1 = l\,(AP) = \sqrt{2}$



$$V_{P2} = \frac{Q_B}{4\pi\varepsilon_0 R_2} \qquad R_2 = l\,(BP) = \sqrt{2}\ m$$

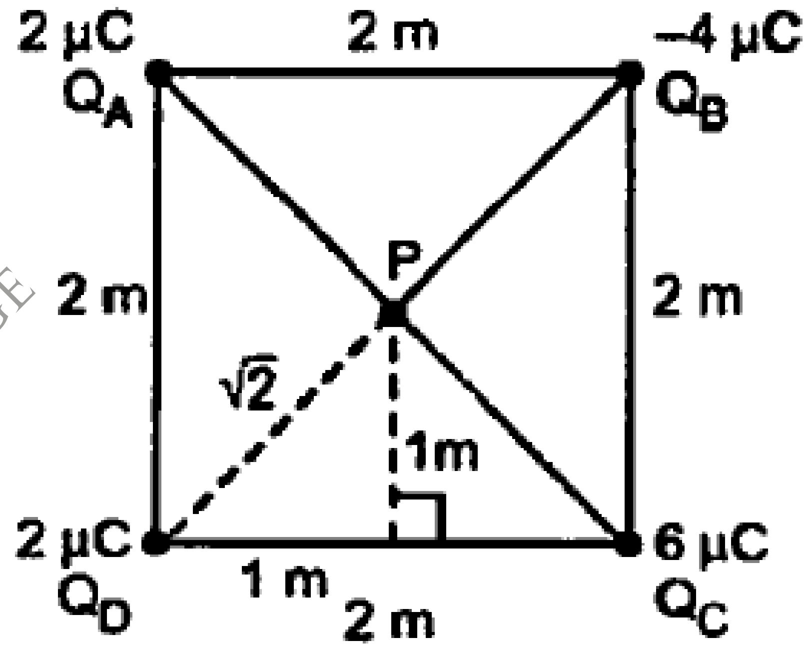
$$V_{P3} = \frac{Q_C}{4\pi\varepsilon_0 R_3} \qquad R_3 = l\,(CP) = \sqrt{2}\ m$$

$$V_{P4} = \frac{Q_D}{4\pi\varepsilon_0 R_4} \qquad R_4 = l\,(DP) = \sqrt{2}\ m$$

$$V_P = \sum_{m=1}^{4} V_{Pm} = \frac{1}{4\pi\varepsilon_0 R}\left[Q_A + Q_B + Q_C + Q_D\right]$$

$$V_P = \frac{1}{4\pi \times 8.854\times10^{-12}\times\sqrt{2}}\,[2-4+6+2]\times10^{-6}$$

$$= 38.131\ kV$$



*Given a field* $\overline{E} = \left(\frac{-6y}{x^2}\right)\bar{a}_x + \left(\frac{6}{x}\right)\bar{a}_y + 5\,\bar{a}_z$ *V/m,*

*Find the potential difference* $V_{AB}$ *given* $A(-7,2,1)$ *and* $B(4,1,2)$.

$$\overline{E} = -\frac{6y}{x^2}\,\bar{a}_x + \frac{6}{x}\,\bar{a}_y + 5\,\bar{a}_z$$

$$V_{AB} = -\int_B^A \overline{E}\cdot\overline{dL} \qquad \text{where} \qquad \overline{dL} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$$

Now $\bar{a}_x \cdot \bar{a}_x = \bar{a}_y \cdot \bar{a}_y = \bar{a}_z \cdot \bar{a}_z = 1$ and all other dot products are zero.

$$\overline{E}\cdot\overline{dL} = -\frac{6y}{x^2}\,dx + \frac{6}{x}\,dy + 5\,dz$$

$$V_{AB} = -\int_B^A -\frac{6y}{x^2}\,dx + \frac{6}{x}\,dy + 5\,dz$$

To obtain the integral as it does not depend on the path from B (4, 1, 2) to A(−7, 2, 1) we can divide the path as,

Path 1, B (4, 1, 2) to (−7, 1, 2) → only x varies, y = 1, z = 2.

Path 2, (−7, 1, 2) to (−7, 2, 2) → only y varies, x = − 7, z = 2.

Path 3, (−7, 2, 2) to A (−7, 2, 1) → only z varies, x = − 7, y = 2.

$$\therefore \qquad V_{AB} = -\left\{ \int_{x=4}^{x=-7} \frac{-6y}{x^2}\, dx + \int_{y=1}^{y=2} \frac{6}{x}\, dy + \int_{z=2}^{z=1} 5\, dz \right\}$$

$$y = 1 \qquad\qquad x = -7$$

$$= -\left\{ -6 \int_{x=4}^{-7} \frac{1}{x^2}\, dx - \frac{6}{7} \int_{y=1}^{2} dy + 5 \int_{z=2}^{1} dz \right\}$$

$$= -\left\{ -6 \left[ -\frac{1}{x} \right]_4^{-7} - \frac{6}{7} [y]_1^2 + 5 [z]_2^1 \right\}$$

$$= -\left\{-6\left[+\frac{1}{7}+\frac{1}{4}\right]-\frac{6}{7}[2-1]+5[1-2]\right\}$$

$$= -\{-2.3571-0.85714-5\}$$

$$= +8.2142 \text{ V}$$

Alternatively find the equations for straight line path from B to A by using,

$$y-y_B = \frac{y_A-y_B}{x_A-x_B}(x-x_B) \quad \text{and} \quad z-z_B = \frac{z_A-z_B}{y_A-y_B}(y-y_B)$$

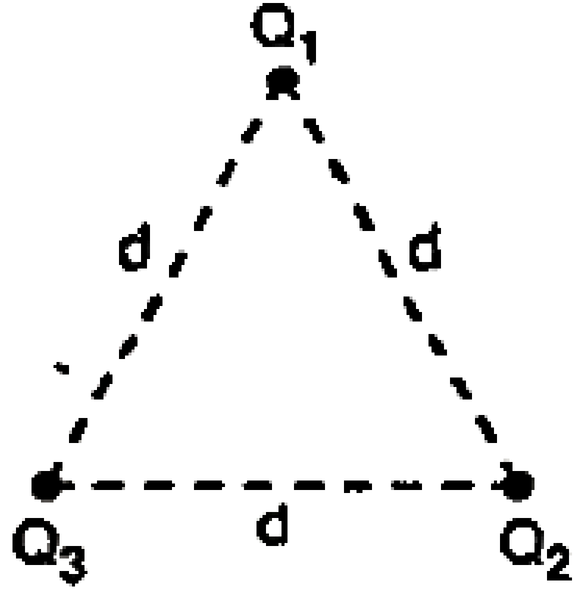
and using the relations between x, y and z solve the integrals. From the above equations we get, $x = -11y + 15$ and $z = -y + 3$ so use y interms of x for first integral and x interms of y for the second integral and integrate.

*Find the potential energy stored in the following free space charge configurations,*

*i) A charge Q at each corner of an equilateral triangle of sides d.*

*ii) A charge Q at each corner of a square of side 'd'.*

*i) A charge Q at each corner of an equilateral triangle of sides d.*




When $Q_1$ is positioned, no other charge is present.

work done $W_1 = 0$ J.

When $Q_2$ is placed, $Q_1$ is present hence work done is,

$$W_2 = Q_2 V_{2,1}$$

$$= \frac{Q_2 Q_1}{4\pi\varepsilon_0 R_{21}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 d}$$

When $Q_3$ is placed, $Q_1$ and $Q_2$ are present hence work done is,

$$W_3 = Q_3 V_{3,1} + Q_3 V_{3,2} = Q_3 \left[ \frac{Q_1}{4\pi\varepsilon_0 R_{31}} + \frac{Q_2}{4\pi\varepsilon_0 R_{23}} \right]$$

Now $\quad R_{31} = R_{23} = d$

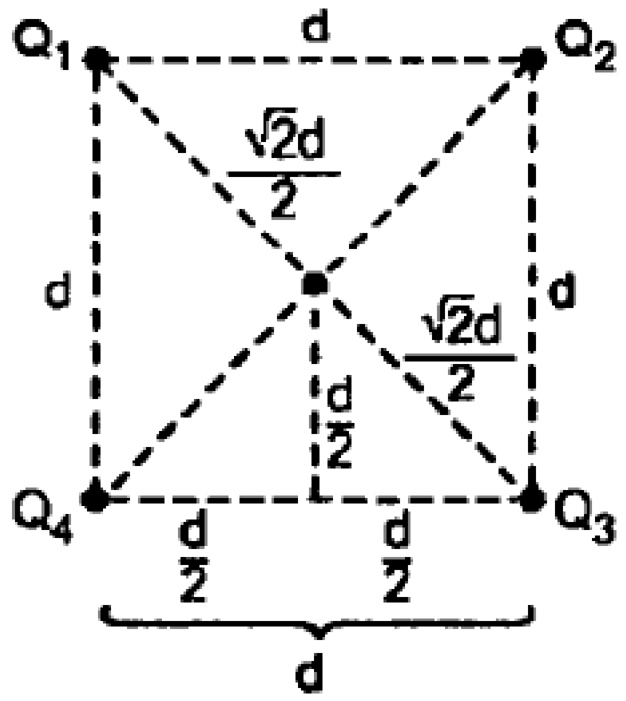
$$\therefore \quad W_3 = \frac{Q_3}{4\pi\varepsilon_0 d}[Q_1 + Q_2]$$

$$\therefore \quad W_E = W_1 + W_2 + W_3 = \frac{1}{4\pi\varepsilon_0 d}[Q_1 Q_2 + Q_1 Q_3 + Q_2 Q_3]$$

but $\quad Q_1 = Q_2 = Q_3 = Q$

$$\therefore \quad W_E = \frac{3Q^2}{4\pi\varepsilon_0 d} \text{ J}$$

*ii) A charge Q at each corner of a square of side 'd'.*




$$R_{12} = d, \quad R_{23} = d, \quad R_{34} = d, \quad R_{41} = d,$$

$$R_{31} = 2\times\frac{\sqrt{2}\,d}{2} = \sqrt{2}\;d = R_{24}$$

For $Q_1$ which is placed first, $W_1 = 0$.

For $Q_2$, $\quad W_2 = Q_2\,V_{2,1}$

$$= \frac{Q_2 Q_1}{4\pi\varepsilon_0 R_{21}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0\, d}$$

For $Q_3$, $W_3 = Q_3\,V_{3,1} + Q_3\,V_{3,2} = Q_3\left[\frac{Q_1}{4\pi\varepsilon_0 R_{31}} + \frac{Q_2}{4\pi\varepsilon_0 R_{32}}\right]$

$$= \frac{Q_1 Q_3}{4\pi\varepsilon_0 \sqrt{2}\,d} + \frac{Q_2 Q_3}{4\pi\varepsilon_0\, d}$$

For $Q_4$, $$W_4 = Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3}$$

$$= Q_4 \left[ \frac{Q_1}{4\pi\varepsilon_0 R_{41}} + \frac{Q_2}{4\pi\varepsilon_0 R_{42}} + \frac{Q_3}{4\pi\varepsilon_0 R_{43}} \right]$$

$$= \frac{Q_1 Q_4}{4\pi\varepsilon_0 d} + \frac{Q_2 Q_4}{4\pi\varepsilon_0 d\sqrt{2}} + \frac{Q_3 Q_4}{4\pi\varepsilon_0 d}$$

And $$Q_1 = Q_2 = Q_3 = Q_4 = Q$$

$$\therefore \quad W_E = W_1 + W_2 + W_3 + W_4 = \frac{Q^2}{4\pi\varepsilon_0 d} \left[ 1 + \frac{1}{\sqrt{2}} + 1 + 1 + \frac{1}{\sqrt{2}} + 1 \right]$$

$$\therefore \quad W_E = \frac{5.414\, Q^2}{4\pi\varepsilon_0 d} \text{ J}$$

# ELECTROMAGNETIC FIELDS
## (19BT30202)

## Electric Dipole & Dipole Moment

## (ELECTROSTATICS – II)

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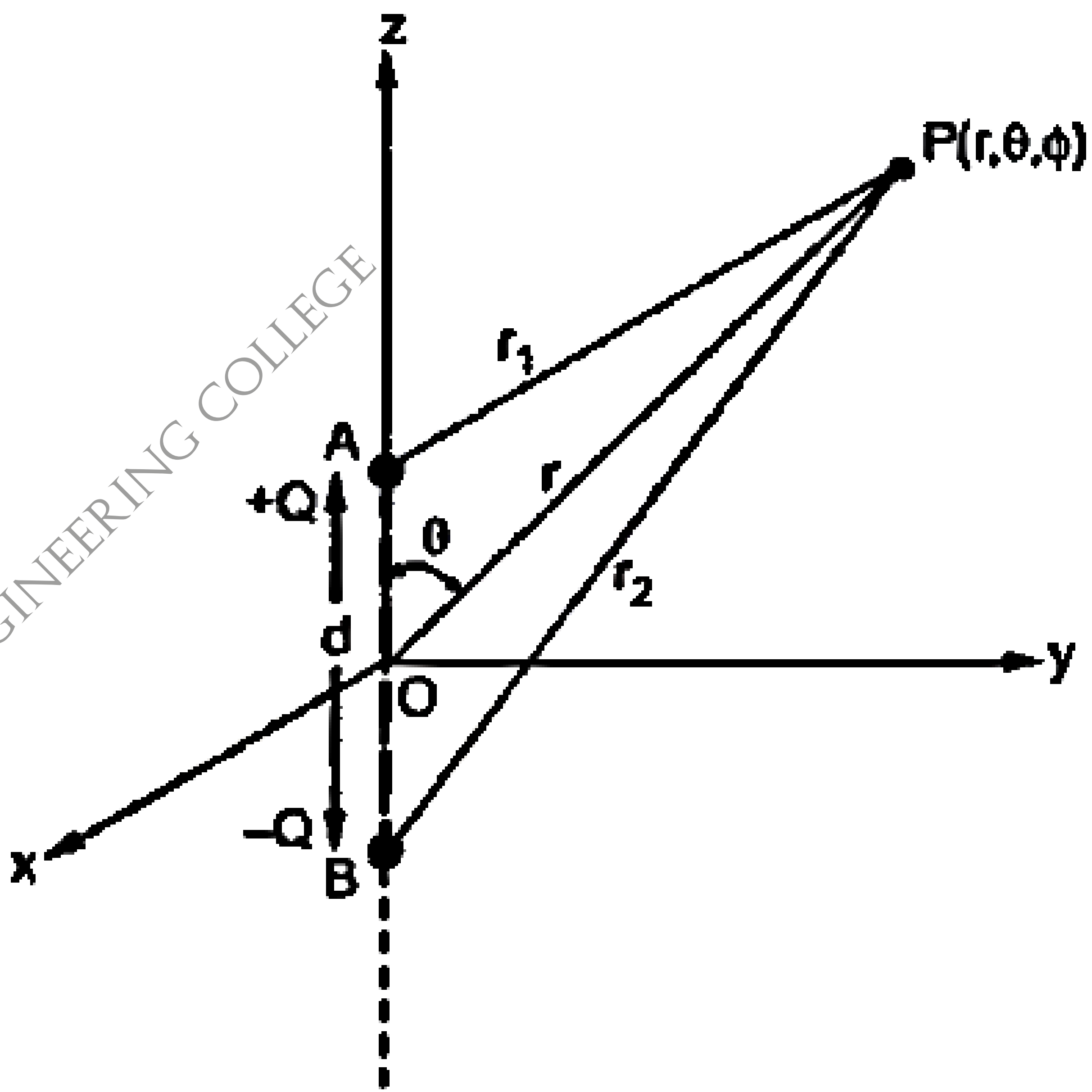
## An Electric Dipole

The two point charges of equal magnitude but opposite sign, seperated by a very small distance give rise to an electric dipole. The field produced by such a dipole plays an important role in the engineering electromagnetics.

Consider an electric dipole as shown in the Fig. 4.32. The two point charges + Q and − Q are separated by a very small distance d.



Consider a point P $(r, \theta, \phi)$ in spherical co-ordinate system. It is required to find $\bar{E}$ due to an electric dipole at point P. Let O be the midpoint of AB. The distance of point P from A is $r_1$ while the distance of point P from B is $r_2$. The distance of point P from point O is r. The distance of separation of charges i.e. d is very small compared to the distances $r_1$, $r_2$ and r. The co-ordinates of A are $\left(0, 0, +\frac{d}{2}\right)$ and that of B are $\left(0, 0, -\frac{d}{2}\right)$.

To find $\bar{E}$, we will find out the potential V at point P, due to an electric dipole. Then using $\bar{E} = -\nabla V$, we can find $\bar{E}$ due to an electric dipole.

## Expression of $\overline{E}$ due to an Electric Dipole

In spherical co-ordinates, the potential at point P due to the charge + Q is given by,

$$V_1 = \frac{+Q}{4\pi\varepsilon_0 r_1}$$

The potential at P due to the charge – Q is given by,

$$V_2 = \frac{-Q}{4\pi\varepsilon_0 r_2}$$

The total potential at point P is the algebraic sum of $V_1$ and $V_2$.

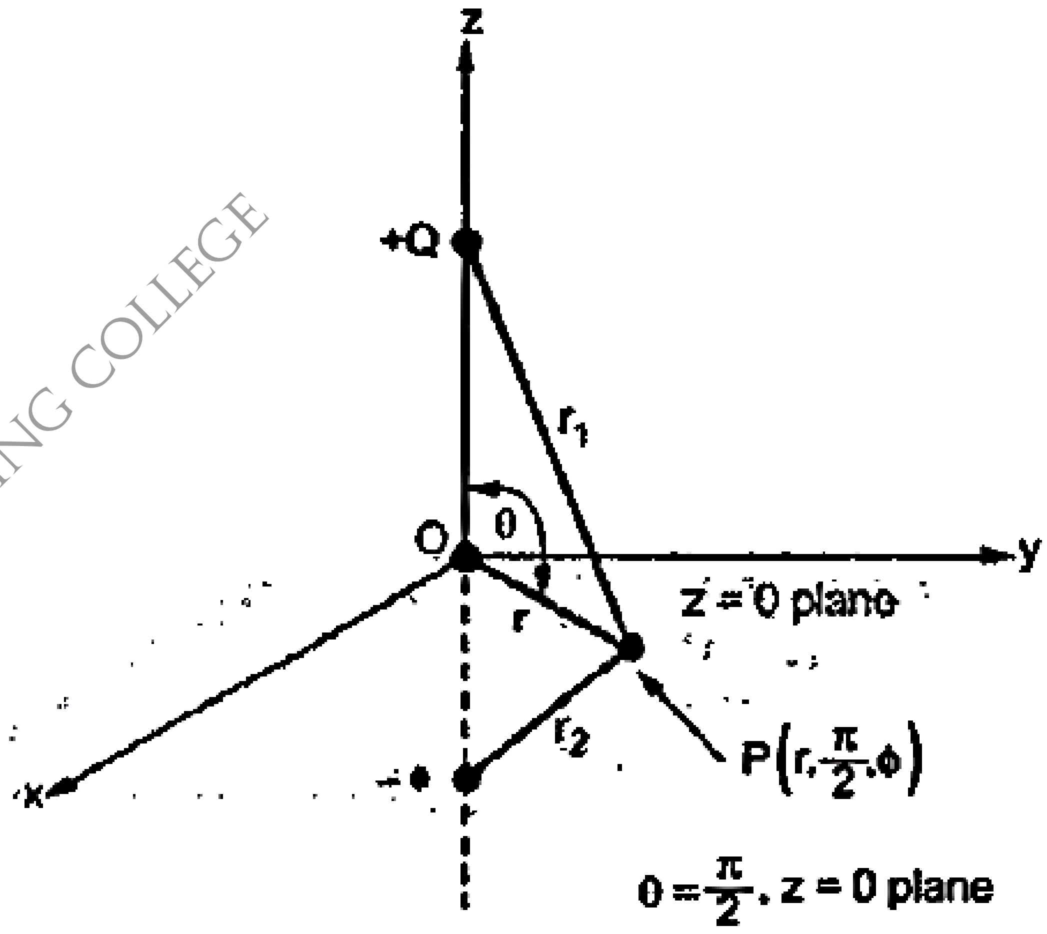
$$\therefore V = V_1 + V_2$$

$$= \frac{+Q}{4\pi\varepsilon_0 r_1} - \frac{Q}{4\pi\varepsilon_0 r_2}$$

$$\therefore V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1\, r_2}\right]$$

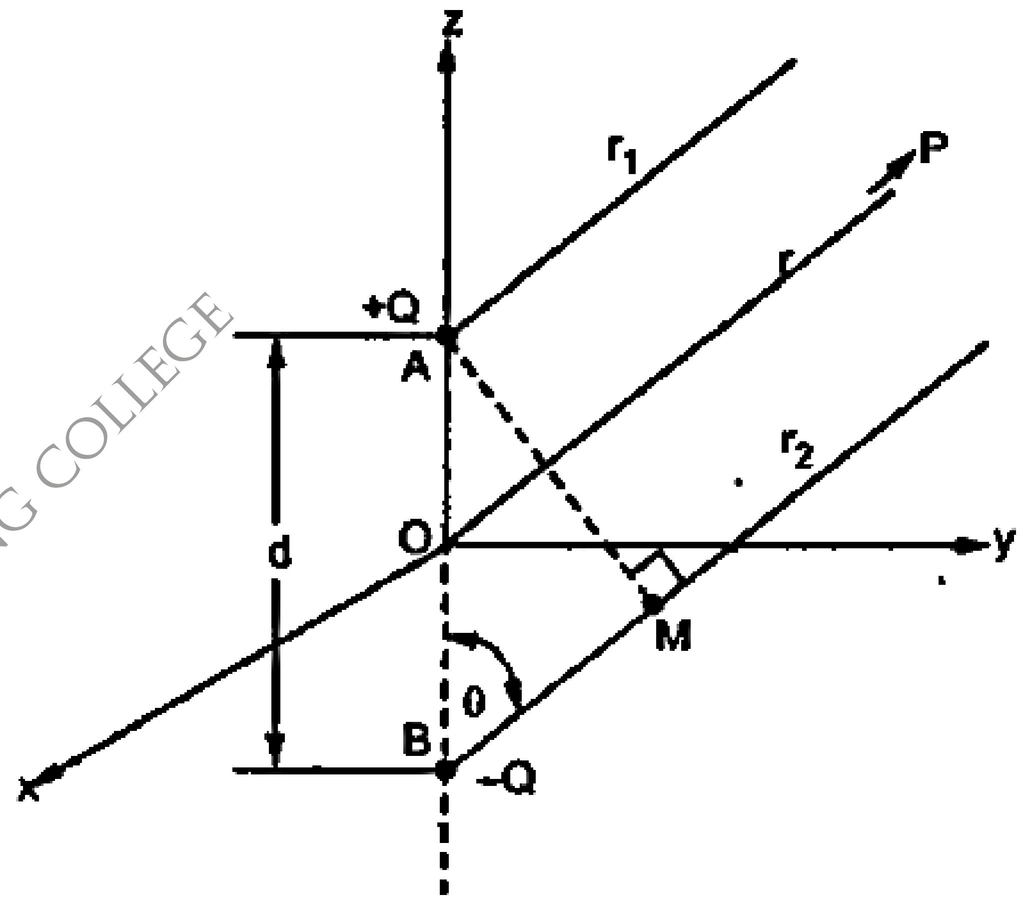
If now point P is located in $z = 0$ plane as shown in the Fig. 4.33, then $r_2 = r_1$. Hence we get $V = 0$. Thus the entire $z = 0$ plane i.e. xy plane is a zero potential surface.

All points in $z = 0$ plane behave similar to the points at infinity as all are at zero potential.



Now consider that P is located far away from the electric dipole. Thus $r_1$, $r_2$ and $r$ can be assumed to be parallel to each other as shown in the Fig. 4.34.

AM is drawn perpendicular from A on $r_2$. The angle made by $r_1$, $r_2$ and $r$ with z axis is $\theta$ as all are parallel.



$$\therefore \quad BM = AB\cos\theta$$

$$= d\cos\theta$$

Now $\quad PB = PM + BM$

and $\quad PA = PM$ as AM is perpendicular.

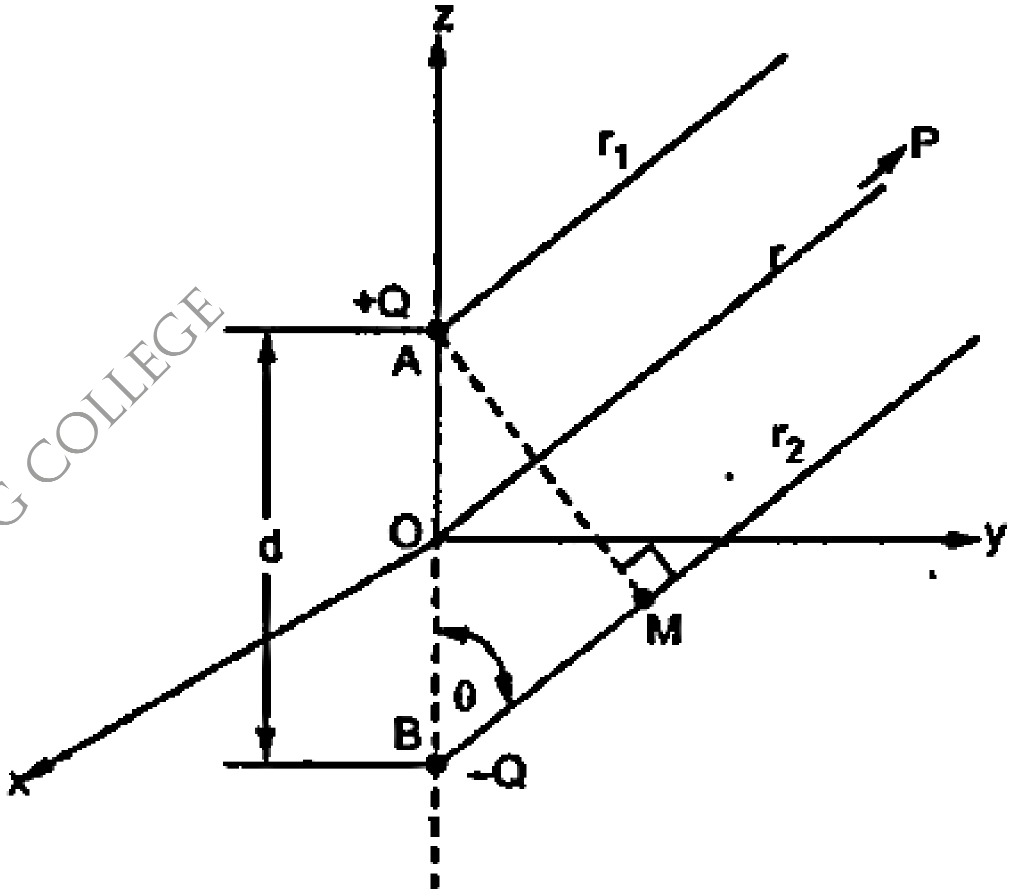
and $\quad PB = r_2, \quad PA = r_1$

$$\therefore \quad BM = PB - PM = r_2 - PM$$

While $\quad PM = PA = r_1$

$$\therefore \quad BM = r_2 - r_1$$

$$\therefore \quad r_2 - r_1 = d\cos\theta$$






$$\therefore V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right]$$

As d is very small, $r_1 \approx r_2 \approx r$ hence $r_1 r_2 = r^2$

$$\therefore \quad V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{d\cos\theta}{r^2}\right] V$$

Now $$\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi\right]$$

$$\therefore \quad \frac{\partial V}{\partial r} = \frac{Q\,d\cos\theta}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)\right] = \frac{Q\,d\cos\theta}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial r}(r^{-2})\right]$$

$$= \frac{Q\,d\cos\theta}{4\pi\varepsilon_0}\left[-2\,r^{-3}\right] = \frac{-2Q\,d\cos\theta}{4\pi\varepsilon_0\,r^3}$$

$$\frac{\partial V}{\partial \theta} = \frac{Q\,d}{4\pi\varepsilon_0\,r^2}\left[-\sin\theta\right] \quad \text{and} \quad \frac{\partial V}{\partial \phi} = 0$$

$$\therefore \quad \bar{E} = -\left[\frac{-2Q\,d\cos\theta}{4\pi\varepsilon_0\,r^3}\bar{a}_r - \frac{Q\,d\sin\theta}{4\pi\varepsilon_0\,r^3}\bar{a}_\theta\right]$$

$$\therefore \quad \boxed{\bar{E} = \frac{Q\,d}{4\pi\varepsilon_0\,r^3}\left[2\cos\theta\,\bar{a}_r + \sin\theta\,\bar{a}_\theta\right]} \quad \text{(Spherical system)}$$

This is electric field $\bar{E}$ at point P due to an electric dipole.

## Dipole Moment

Let the vector length directed from $-Q$ to $+Q$ i.e. from B to A is $\bar{d}$.

$$\therefore \quad \bar{d} = d\,\bar{a}_z$$

Its component along $\bar{a}_r$ direction can be obtained as,

$$d_r = \bar{d} \cdot \bar{a}_r = d\bar{a}_z \cdot \bar{a}_r = d\cos\theta$$

$$\therefore \quad \bar{d} = d\cos\theta\,\bar{a}_r$$

Then the product $Q\,\bar{d}$ is called **dipole moment** and denoted as $\bar{p}$.

$$\therefore \quad \boxed{\bar{p} = Q\,\bar{d}}$$

The dipole moment is measured in Cm (coulomb-metre).

Now $\quad \bar{p} \cdot \bar{a}_r = Q\,\bar{d} \cdot \bar{a}_r = Q\,d\cos\theta$

Hence the expression of potential V can be expressed as,

$$\boxed{V = \frac{Q\,d\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\bar{p} \cdot \bar{a}_r}{4\pi\varepsilon_0 r^2}\ \mathrm{V}}$$

# UNIT - II
# CONDUCTORS AND DIELECTRICS

# UNIT-II CONDUCTORS, DIELECTRICS AND CAPACITANCE

The flow of charge per unit time i.e, rate of flow of charge at a specified point or across a specified surface is called an Electric current. (The current which exists in the conductor due to the drifting of electrons under the influence of the applied voltage is called drift current)

$$I = \frac{dQ}{dt} \quad C/s \text{ (or) Amp}$$

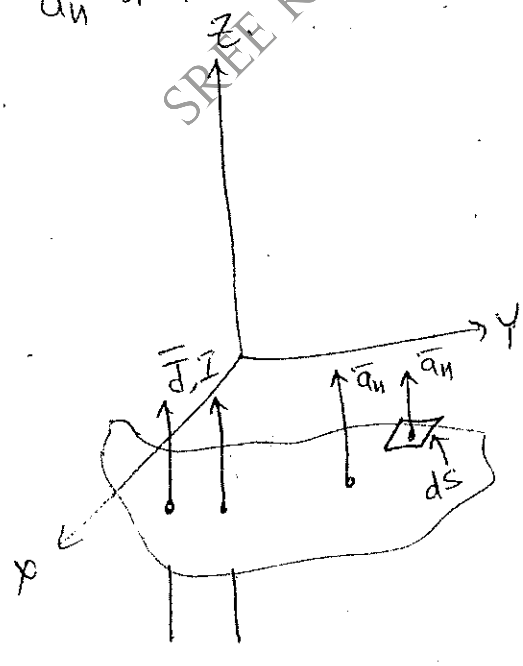
A current of 1 Amp is said to be flowing across the surface when a charge of one coulomb is passing across the surface in one second. (while in dielectrics, there can be flow of charge under the influence of $\bar{E}$ such a current is called displacement current or convection current Eg: capacitor)

## CURRENT DENSITY (J):-

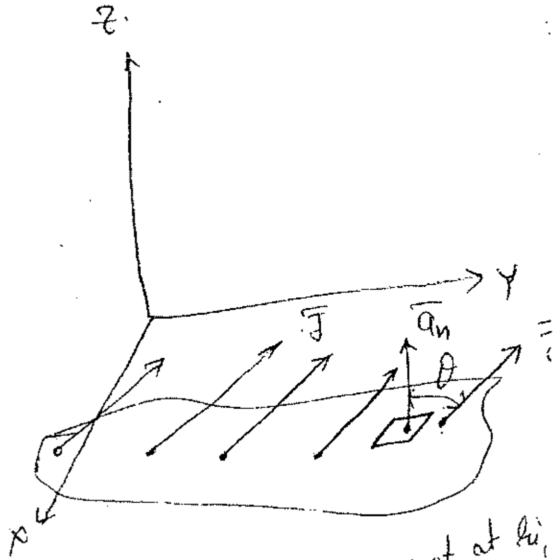
It is defined as the current passing through the unit surface area, when the surface is held normal to the direction of the current. (A/m²)

Consider a surface S and I is the current passing through the surface. The direction of current is normal to the surface S and hence direction of $\bar{J}$ is also normal to the surface S.

Consider an incremental surface area dS as shown in, and $\bar{a}_n$ is the unit vector normal to the incremental surface



$\bar{J}$ & $d\bar{S}$ are normal



$\bar{J}$ & $d\bar{S}$ are not at right angle

$$d\bar{S} = ds\,\bar{a}_n$$

$$\bar{J} = J\,\bar{a}_n$$

$dI$ passing through the $ds$ is given by

$$dI = \bar{J} \cdot d\bar{S}$$

when $\bar{J}$ & $d\bar{S}$ are at right angles ($\theta = 90°$) then

$$dI = J\bar{a}_n \cdot ds\,\bar{a}_n$$

$$= J\,ds$$

$$I = \int_S J\,ds$$

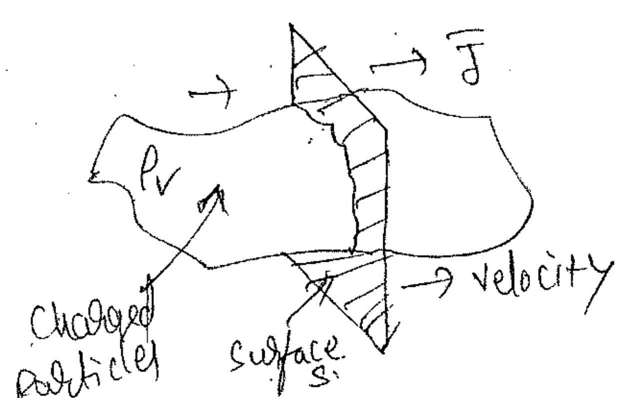
If $\bar{J}$ is not normal to $d\bar{S}$ then

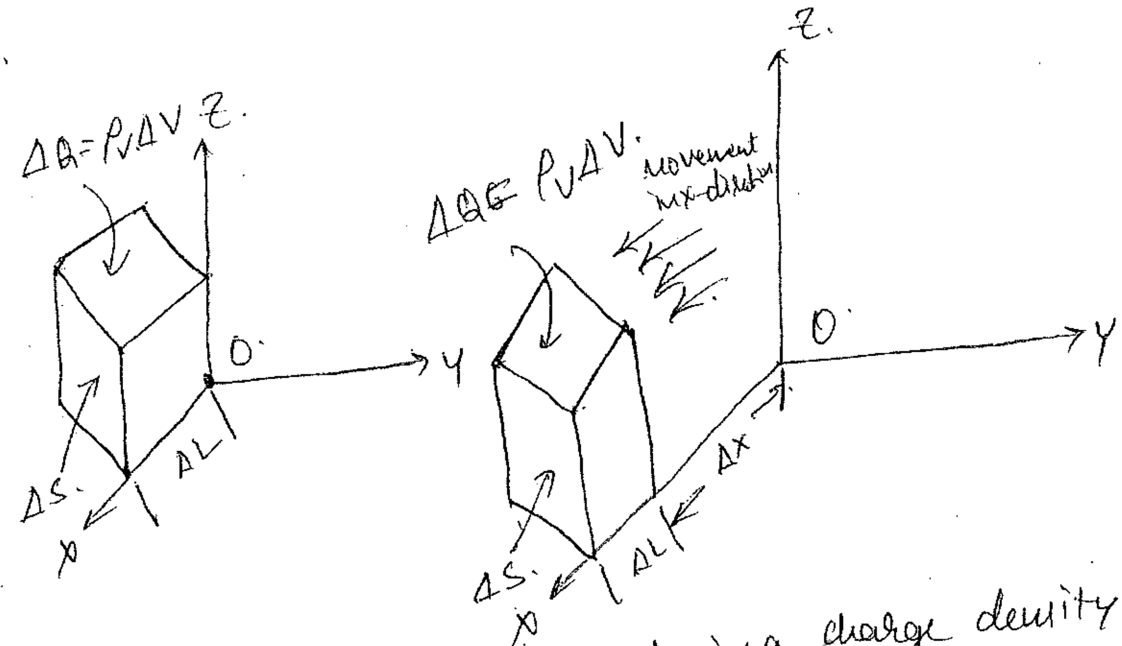
$$I = \int_S \bar{J} \cdot d\bar{S} \quad \text{(dot product)}.$$

RELATION BETWEEN $\bar{J}$ & $\rho_v$ :-

The set of charged particles give rise to a charge density $\rho_v$ in a volume V.

The current density $\bar{J}$ can be related to the velocity with which the volume charge i.e., charged particles in volume V crosses the surface S at a point.






Consider a differential volume $\Delta V$ having charge density $\rho_V$. That volume carries elementary charge

$$\Delta Q = \rho_V \Delta V$$

Let $\Delta L$ is the incremental length while $\Delta S$ is the incremental surface area hence

$$\Delta V = \Delta S \, \Delta L$$

$$\Delta Q = \rho_V \, \Delta S \, \Delta L$$

Let the charge is moving in X direction with velocity $\bar{V}$ and thus velocity has only X component $V_x$.

In the time interval $\Delta t$ the element of charge has moved through distance $\Delta X$, in X-direction.

$$\Delta I = \frac{\Delta Q}{\Delta t}$$

$$\Delta I = \rho_V \, \Delta S \, \frac{\Delta X}{\Delta t} \qquad (\because \Delta L = \Delta X).$$

$$\frac{\Delta X}{\Delta t} = \text{velocity}$$

$$\therefore \Delta I = \rho_V \, \Delta S \, V_x.$$

but $\Delta I = \bar{J} \Delta S$ when $\bar{J}$ & $\Delta S$ are normal.

Comparing the two equations.

$$J_x = \rho_V V_x = x \text{ component of } \bar{J}$$

in general, the relation b/w $\bar{J}$ & $\rho_V$

$$\Rightarrow \boxed{\bar{J} = \rho_V \bar{V}}$$

Such a current is called convection current & the current density is called convection current density.

CONTINUITY EQUATION:-

The continuity equation of the current is based on the principle of conservation of charge.

The charges can neither be created nor be destroyed.

Consider a closed surface S with a current density $\bar{J}$ then the total current $I$ crossing the surface S is given by

$$I = \oint_S \bar{J} \cdot d\bar{S}$$

The current flows outwards from the closed surface. Hence the current means the flow of positive charges. Hence the $I$ is constituted due to outward flow of +ve charges from the closed surface S.

According to principle of conservation of charge, there must be decrease of an equal amount of +ve charge inside the closed surface.

Hence the outward rate of flow of +ve charge gets balanced by the rate of decrease of charge inside the closed surface.

Let $Q_i$ = charge within the closed surface (3)

$-\frac{dQ_i}{dt}$ = Rate of decrease of charge inside the closed surface

$$I = \oint_S \bar{J}\cdot d\bar{S} = -\frac{dQ_i}{dt}$$

This is the integral form of the continuity equation of the current.

If the current is entering the volume then

$$\oint_S \bar{J}\cdot d\bar{S} = -I = \frac{dQ_i}{dt}$$

⇒ The point form can be obtained from the integral form using the divergence theorem,

$$\oint_S \bar{J}\cdot d\bar{S} = \int_{vol} (\nabla\cdot\bar{J})\,dV$$

$$-\frac{dQ_i}{dt} = \int_{vol} (\nabla\cdot\bar{J})\,dV$$

$$Q_i = \int_{vol} \rho_v\,dv$$

$$\therefore \int_{vol} (\nabla\cdot\bar{J})\,dV = -\frac{d}{dt}\left[\int_{vol} \rho_v\,dv\right]$$

$$= -\int_{vol} \frac{\partial \rho_v}{\partial t}\,dv.$$

For a constant surface,

$$\int_{vol} (\nabla\cdot\bar{J})\,dV = \int_{vol} -\frac{\partial \rho_v}{\partial t}\,dV$$

for incremental volume $\Delta V$

$$(\nabla \cdot \bar{J})\Delta V = -\frac{\partial \rho_v}{\partial t}\Delta V$$

$$\boxed{\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}}$$

This is the point form or differential form of the continuity equation of the current.

The equation states that the current or the charge per unit volume per sec. diverging from a small volume per unit volume is equal to the time rate of decrease of charge/unit volume at every point.

for steady currents which are not the functions of time, $\frac{\partial \rho_v}{\partial t} = 0$. hence, $\boxed{\nabla \cdot \bar{J} = 0}$



* Find the total current in outward direction from a cube of 1m, with one corner at the origin and edges parallel to the co-ordinate axes if,

$$\bar{J} = 2x^2\bar{a}_x + 2xy^3\bar{a}_y + 2xy\bar{a}_z \ A/m^2$$

Sol:-
$$I = \oint_S \bar{J}\cdot d\bar{s} = \int_{vol} (\nabla \cdot \bar{J})\, dv.$$

$dv = dx\,dy\,dz.$

$$\nabla \cdot \bar{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}$$

$$= 4x + 6xy^2$$

$$I = \underline{3A.}$$

## CONDUCTORS:-

Under the effect of applied electric field, the available free electrons start moving. The moving electrons strike the adjacent atoms and rebound in the random direction. This is called drifting of electrons. After some time, the electrons attain the const. avg velocity called drift velocity ($V_d$). The current due to the drifting of such electrons in metallic conductors is called drift current.

The drift velocity depends on the applied electric field

$$\therefore \bar{V_d} \propto \bar{E}$$

$$\bar{V_d} = -\mu_e \bar{E}$$

$\mu_e$ - mobility of electrons

-ve - velocity of electrons is against the $\bar{E}$

we know that

$$\bar{J} = \rho_V \bar{V}$$

but in the material, no. of protons = no. of electrons & it is always electrically neutral, hence $\rho_V = 0$ for neutral material.

$$\therefore \bar{J} = \rho_e \bar{V_d}, \quad \rho_e = \text{charge density due to free electrons}$$

$$\rho_e = ne$$

$n \rightarrow$ no. of free electrons $/m^3$

$e \rightarrow$ charge on one electron.

$$\boxed{\therefore \bar{J} = -\rho_e \mu_e \bar{E}}$$

## POINT FORM OF OHM'S LAW:-

The relation b/w $\bar{J}$ & $\bar{E}$ can be expressed in terms of conductivity of the material.

Thus for a metallic conductor, $\boxed{\bar{J} = \sigma \bar{E}}$

$\sigma$ - conductivity of the material $\mho/m$

The above equation is called point form of ohm's law.

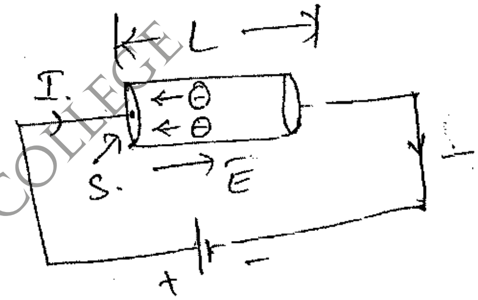
we know that $\bar{J} = -\rho_e \mu_e \bar{E}$

& $\bar{J} = \sigma \bar{E}$

$\therefore \sigma = -\rho_e \mu_e$

Resistivity is the reciprocal of the conductivity. The conductivity depends on the temp. As the temp ↑ the vibrations of crystalline structure of atoms increases, due to increased vibrations of electron drift velocity decreases, hence the mobility & conductivity decreases

RESISTANCE OF A CONDUCTOR:-




V - applied voltage

L - conductor length

S - uniform cross-section

$\bar{E}$ - direction is conventional current direction

$\bar{E}$ applied is uniform

$$E = \frac{V}{L}$$

Conductor has uniform cross-section S.

$$I = \int_S \bar{J} \cdot d\bar{S} = JS$$

current direction is normal to the surface S.

$$J = \frac{I}{S} = \sigma E$$

$$J = \frac{\sigma V}{L}$$

$$V = \frac{JL}{\sigma}$$

$$= \frac{IL}{\sigma S}$$

$$= \left(\frac{L}{\sigma S}\right) I$$

define R. $\boxed{R = \frac{V}{I} = \frac{L}{\sigma S}}$

for non-uniform fields.

$$R = \frac{V_{ab}}{I} = \frac{-\int_b^a \bar{E} \cdot d\bar{L}}{\int_S \bar{J} \cdot d\bar{S}}$$

$$R = \frac{-\int_b^a \bar{E} \cdot d\bar{L}}{\int_S \sigma \bar{E} \cdot d\bar{S}}$$

## PROPERTIES OF CONDUCTOR:-

(1) under static condition, no charge & no electric field can exist at any point within the conducting material

(2) The charge can exist on the surface of the conductor giving rise to surface charge density.

(3) Within a conductor, the charge density is always zero

(4) The charge distribution on the surface depends on the shape of the surface.

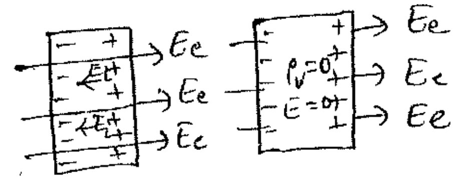
(5) The conductivity of an ideal conductor is infinite

(6) The conductor surface is an equipotential surface.

## RELAXATION TIME:-

The medium is called homogeneous when the physical char. of the medium do not vary from point to point but remain same every where through out the medium

— nonhomogeneous (or) heterogeneous




Linear → $\bar{D} \propto \bar{E}$

non-linear → If $\bar{D}$ is not $\propto \bar{E}$

Consider a conducting material which is linear & homogeneous.

when an $E_c$ is applied, the $+^{ve}$ free charges are pushed along the same direction as the applied field, while the negative free charges move in the opposite direction. They accumulate on the surface of the conductor and form surface charges. The induced charges set up an internal induced field $E_i$ which cancels the applied $E_c$.

$$\bar{J} = \sigma \bar{E}$$

$$\bar{D} = \varepsilon \bar{E}$$

$$\bar{E} = \frac{\bar{D}}{\varepsilon}$$

$$\bar{J} = \sigma \frac{\bar{D}}{\varepsilon} = \frac{\sigma}{\varepsilon} \bar{D}$$

point form of continuity eqn.

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$$

$$\nabla \cdot \left(\frac{\sigma}{\varepsilon} \bar{D}\right) = -\frac{\partial \rho_v}{\partial t}$$

$$\frac{\sigma}{\varepsilon}\nabla\cdot\bar{D} = -\frac{\partial \rho_v}{\partial t}$$

$$\nabla\cdot\bar{D} = \rho_v$$

$$\frac{\sigma \rho_v}{\varepsilon} = -\frac{\partial \rho_v}{\partial t}$$

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon}\rho_v = 0$$

This is a differential eqn. in $\rho_v$ whose solution is

$$\rho_v = \rho_0 e^{-(\sigma/\varepsilon)t}$$

$$= \rho_0 e^{-t/\tau}$$

$\rho_0$ = charge density at $(t=0)$

This shows that if there is a temporary imbalance of electrons inside the given material, the charge density decays exponentially with a time constant $\tau = \frac{\varepsilon}{\sigma}$ sec. This time is called relaxation time.

DIELECTRIC MATERIALS:-

→ do not have free charges.

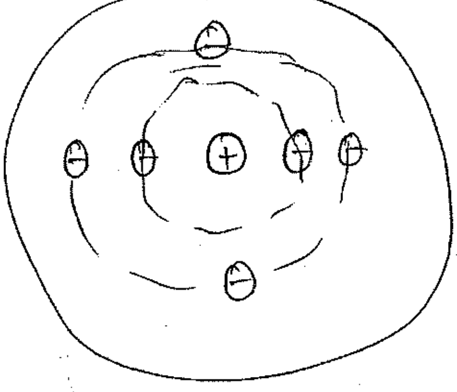
→ bound charges: The charges in dielectrics are bound by the finite forces & hence called bound charges

→ when the dipole results from the displacement of the bound charges, the dielectric is said to be polarized.

POLARIZATION:-

$\bar{E} = 0$: these dipoles produce an electric field which opposes the external applied field. This process due to which seperation of bound charges results to produce

no. of +ve charges = no. of −ve charges



hence atom is electrically neutral

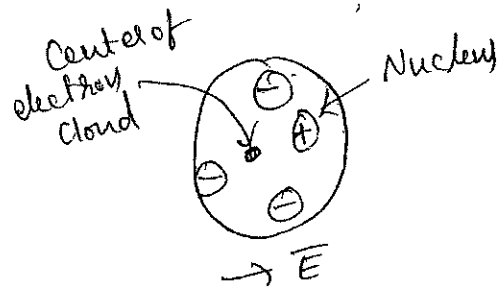
$\bar{E} \neq 0$ electric dipoles, under the influence of $\bar{E}$ is called polarization.

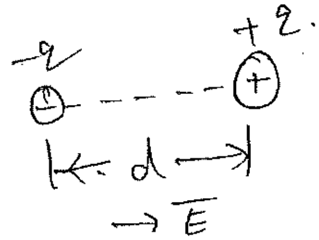
when $\bar{E}$ is applied, the symmetrical distribution of charges gets disturbed. ③

The $+^{ve}$ charges experience a force $\bar{F} = Q\bar{E}$

$$\bar{F} = -Q\bar{E}$$

such an atom is called polarised atom.




The electron cloud has a centre separated from the nucleus. It is an electric dipole.

Two types of dielectrics:
- Non-polar – dipole totally absent when $E$ is applied dipole exists
- polar – dipole exists dipoles are randomly a

MATHEMATICAL EXPRESSION FOR POLARIZATION:-

when the dipole is formed due to polarization, there exists an electric dipole moment $\bar{P}$

$$\bar{P} = Q\bar{d}$$

$Q$ = magnitude of one of the 2 charges

$\bar{d}$ = distance vector from $-^{ve}$ to $+^{ve}$ charge.

Let $n$ = no. of dipoles / unit volume

$\Delta V$ = total volume of the dielectric

$N$ = Total dipoles = $n\Delta V$

$$\bar{P}_{total} = Q_1\bar{d_1} + Q_2\bar{d_2} + \cdots Q_n\bar{d_n} = \sum_{i=1}^{n\Delta V} Q_i\bar{d_i}$$

If dipoles are randomly oriented, $\bar{P}_{total} = 0$

but if " " aligned in the direction of $E$ the $\bar{P}_{total}$ has some val.

$$\bar{P} = \lim_{\Delta V \to 0} \frac{\sum_{i=1}^{n\Delta V} Q_i\bar{d_i}}{\Delta V} \quad C/m^2$$

unit of polarization are same as that of $\bar{D}$.
thus polarization increases the $\bar{D}$ in a dielectric medium.

$$\bar{D} = \varepsilon_0 \bar{E} + \bar{P}$$

for isotropic & linear medium, the $\bar{P}$ & $\bar{E}$ are parallel to each other at every point & related to each other as,

$$\bar{P} = X_e \varepsilon_0 \bar{E}$$

$X_e$ = dimensionless quantity called electric susceptibility of the material

$$\bar{D} = \varepsilon_0 \bar{E} + X_e \varepsilon_0 \bar{E}$$

$$\bar{D} = (X_e + 1) \varepsilon_0 \bar{E}$$

$$\bar{D} = \varepsilon \bar{E}$$

$$\varepsilon = \varepsilon_R \varepsilon_0$$

the quantity $X_e + 1$ is defined as relative permittivity or dielectric constant of the dielectric material.

$$\varepsilon_R = X_e + 1$$

while the $\varepsilon$ is called permittivity of the dielectric.

PROPERTIES OF DIELECTRIC MATERIALS:-

(1) dielectrics do not contain any free charges but contain bound charges.
(2) Bound charges are under the internal molecular & atomic forces & can not contribute to the conduction.
(3) when subjected to an $\bar{E}$, the bound charges shift their relative position. due to this, small ~~d~~ electric dipoles gets induced inside the dielectric. This is called polarization.

(4) due to the polarization, the dielectrics can store the energy (4)

(5) Due " " the flux density of the dielectric increases by amount equal to the polarization.

(6) The induced dipoles produce their own electric field & align in the direction of the applied electric field.

(7) when polarization occurs, the volume charge density is formed inside the dielectric while the surface charge density is formed over the surface of the dielectric.

(8) The electric field outside & inside the dielectric gets modified due to the induced electric dipoles.

BOUNDARY CONDITIONS:-

The conditions existing at the boundary of the two media when field passes from one medium to other are called boundary conditions.

Depending upon the nature of the media, there are two situations of the boundary conditions.

(1) Boundary b/w conductor and free space

(2) Boundary b/w two dielectrics with different properties

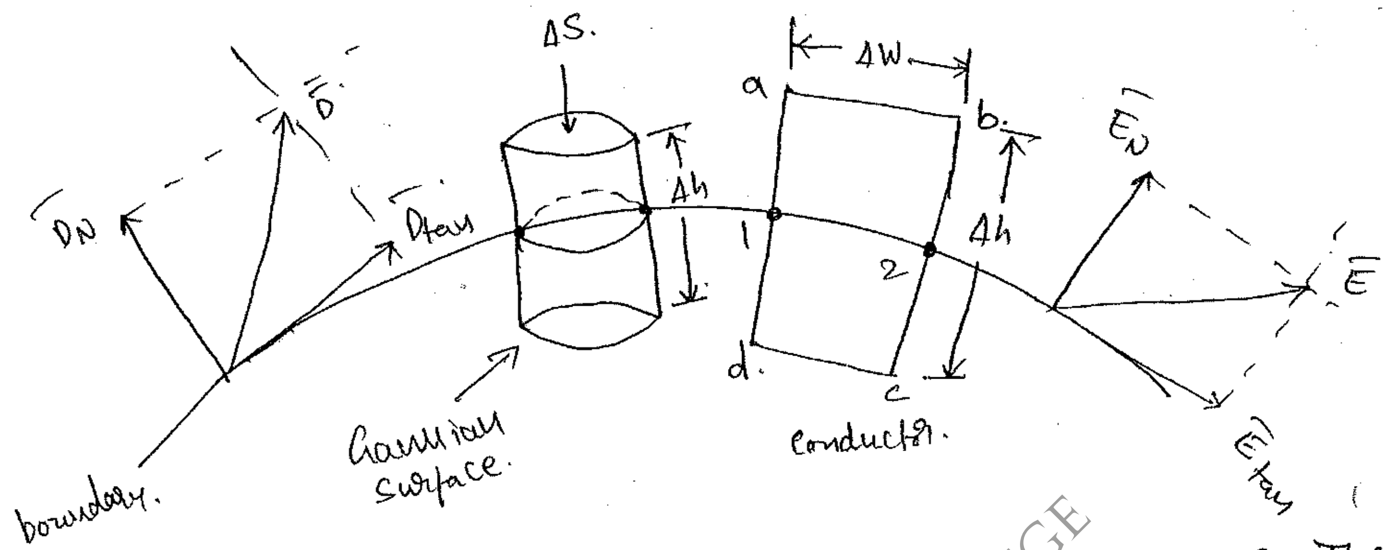
the free space is nothing but a dielectric hence I case is nothing but the boundary b/w cond & dielectr for studying the boundary conditions, the maxwell's equati are required.

$$\oint \bar{E} \cdot \bar{dL} = 0$$

$$\oint \bar{D} \cdot \bar{ds} = Q$$

III$^{ly}$ $\bar{E}$ is required to be decomposed into two compone namely tangential to the boundary ($\bar{E}_{tan}$) & normal to boundary ($\bar{E}_N$). $\boxed{\bar{E} = \bar{E}_{tan} + \bar{E}_N}$ III$^{ly}$ $\bar{D} = \bar{D}_{tan} + \bar{D}_N$

BOUNDARY CONDITIONS BETWEEN CONDUCTOR AND FREE SPACE:-




Consider a boundary b/w conductor & free space. The conductor is ideal having infinite conductivity eg. cu, silver etc,

(1) The $\bar{E}$ inside a conductor is zero & also the $\bar{D}$

(2) No charge can exist within a conductor. The charge appear on the surface in the form of surface charge density.

(3) The charge density within the conductor is zero.

$\bar{E}$ AT THE BOUNDARY:-

Let $\bar{E}$ in the direction as shown in fig, making some angle with the boundary. This E can be resolved into 2 components:

(1) The component tangential to the surface ($\bar{E}_{tan}$)

(2) The component normal to the surface ($\bar{E}_N$)

We know $\oint \bar{E} \cdot d\bar{l} = 0$. i.e, workdone in carrying unit +ve charge along a closed path is zero.

Consider a rectangular closed path abcda (5)
it is traced in clockwise direction as a-b-c-d-a
& hence $\oint \bar{E}\cdot d\bar{l}$ can be divided into.

$$\oint \bar{E}\cdot d\bar{l} = \int_a^b \bar{E}\cdot d\bar{l} + \int_b^c \bar{E}\cdot d\bar{l} + \int_c^d \bar{E}\cdot d\bar{l} + \int_d^a \bar{E}\cdot d\bar{l} = 0.$$

The closed contour is placed in such a way that its 2 sides a-b & c-d are parallel to tangential direction to the surface while the other two are normal to the surface, at the boundary.

$\Delta h$ –.
$\Delta w$ –

$\frac{\Delta h}{2}$ is in the conductor
$\frac{\Delta h}{2}$ is in the free space

Now the portion c-d is in the conductor where $\bar{E}=0$
hence the corresponding integral is zero.

$$\therefore \int_a^b \bar{E}\cdot d\bar{l} + \int_b^c \bar{E}\cdot d\bar{l} + \int_d^a \bar{E}\cdot d\bar{l} = 0.$$

$\because$ $\Delta w$ is very small, $\bar{E}$ over it can be assumed constant

$$\int_a^b \bar{E}\cdot d\bar{l} = \bar{E}\int_a^b d\bar{l} = \bar{E}(\Delta w)$$

but $\Delta w$ is along tangential direction to the boundary in which direction $\bar{E} = \bar{E}_{tan}$

$$\therefore \int_a^b \bar{E}\cdot d\bar{l} = E_{tan}(\Delta w)$$

$$E_{tan} = |\bar{E}_{tan}|$$

now b-c is parallel to normal component

$$\bar{E} = \bar{E}_N$$

$$\therefore \int_b^c \bar{E} \cdot d\bar{L} = E \int_b^c d\bar{L} = E_N \int_b^c d\bar{L}.$$

but out of b-c, b-2 is in free space & 2-c is in the conductor

$$\int_b^c d\bar{L} = \int_b^2 d\bar{L} + \int_2^c d\bar{L} = \frac{\Delta h}{2} + 0 = \frac{\Delta h}{2}$$

$$\int_b^c \bar{E} \cdot d\bar{L} = E_N \left( \frac{\Delta h}{2} \right).$$

||| for d-a, = path b-c, only direction is opposite.

$$\int_d^a \bar{E} \cdot d\bar{L} = -E_N \left( \frac{\Delta h}{2} \right).$$

$$\therefore E_{tan} \Delta W + E_N \left( \frac{\Delta h}{2} \right) - E_N \left( \frac{\Delta h}{2} \right) = 0.$$

$$E_{tan} \Delta W = 0.$$ but $\Delta W \neq 0$ as finite

$$\boxed{\therefore E_{tan} = 0.}$$

Thus the tangential component of the $\bar{E}$ is zero at the boundary b/w conductor & free space.

(*) Thus the $\bar{E}$ at the boundary b/w conductor & free space is always in the direction $\perp^r$ to the boundary.

$$\bar{D} = \varepsilon_0 \bar{E}$$

$$D_{tan} = \varepsilon_0 E_{tan} = 0$$

$$\boxed{\therefore D_{tan} = 0}$$

$D_N$ AT THE BOUNDARY:-

to find normal component of $\bar{D}$, select a closed gaussian surface in the form of right circular cylinder as shown in fig. its height $\Delta h$ & is placed in such a way that $\Delta h/2$ is in the conductor & remaining $\Delta h/2$ is in the free space.

$$\oint_S \bar{D}\cdot d\bar{S} = Q$$

$$\therefore \int_{top} \bar{D}\cdot d\bar{S} + \int_{bottom} \bar{D}\cdot d\bar{S} + \int_{lateral} \bar{D}\cdot d\bar{S} = Q.$$

bottom surface $\bar{D}=0 \quad \therefore \int_{bottom} = 0.$

$$\therefore \int_{top} \bar{D}\cdot d\bar{S} + \int_{lateral} \bar{D}\cdot d\bar{S} = Q,$$

The top surface is in the free space and we are interested in the boundary condition hence top surface can be shifted at the boundary with $\Delta h$.

$$\therefore \int_{top} \bar{D}\cdot d\bar{S} = D_N \int_{top} d\bar{S}$$

$$= D_N \Delta S.$$

The lateral surface area is $2\pi r \Delta h$ where $r$ is radius but as $\Delta h \to 0$, this area reduces to zero & corresponding integ

$$\therefore D_N \Delta S = Q.$$

but at the boundary, the charge exists in the form of $\rho_s$

$$\therefore Q = \rho_s \Delta S.$$

$$\therefore D_N \Delta S = \rho_s \Delta S$$

$$\boxed{\therefore D_N = \rho_s}$$

Thus the flux leaves the surface normally & the normal component of flux density equal to the surface charge den

$$D_N = \varepsilon_0 E_N = \rho_s$$

$$\boxed{E_N = \frac{\rho_s}{\varepsilon_0}}$$

## BOUNDARY CONDITIONS B/W CONDUCTOR & DIELECTRIC:-

The free space is a dielectric with $\varepsilon = \varepsilon_0$.
Then if the boundary is b/w conductor & dielectric with $\varepsilon = \varepsilon_0 \varepsilon_r$

$$E_{tan} = D_{tan} = 0$$

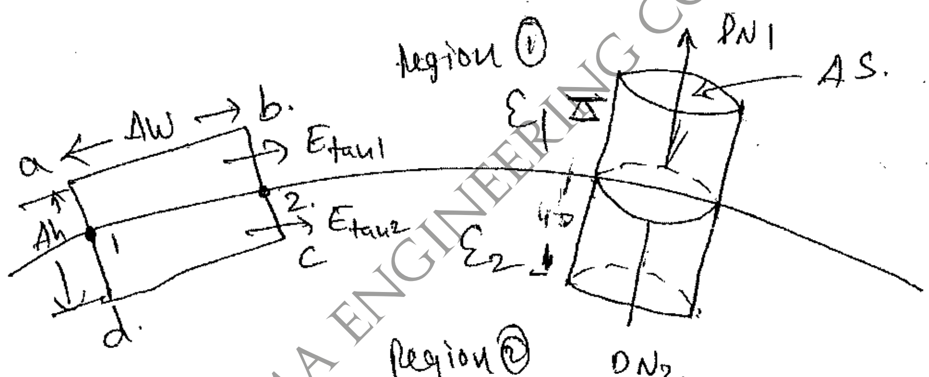
$$D_N = \rho_S$$

$$E_N = \frac{\rho_S}{\varepsilon} = \frac{\rho_S}{\varepsilon_0 \varepsilon_r}$$

## BOUNDARY CONDITIONS B/W TWO PERFECT DIELECTRICS:-

→ one dielectric has $\varepsilon_1$

→ the other has $\varepsilon_2$






Consider a closed path abcda rectangular in shape having elementary height Δh & elementary width ΔW. It is placed in such a way that $\frac{\Delta h}{2}$ is in dielectric 1 & remaining in 2

a-b-c-d-a. $\oint \bar{E} \cdot d\bar{l} = 0$

$$\therefore \int_a^b \bar{E} \cdot d\bar{l} + \int_b^c \bar{E} \cdot d\bar{l} + \int_c^d \bar{E} \cdot d\bar{l} + \int_d^a \bar{E} \cdot d\bar{l} = 0.$$

$\bar{E}_1 = \bar{E}_{1t} + \bar{E}_{1N}$

$\bar{E}_2 = \bar{E}_{2t} + \bar{E}_{2N}$

Let $|\bar{E}_{1t}| = E_{tan1}$, $|\bar{E}_{2t}| = E_{tan2}$

$|\bar{E}_{1N}| = E_{1N}$, $|\bar{E}_{2N}| = E_{2N}$

Now for the rectangle to be reduced at the surface to analyse boundary conditions, $\Delta h \to 0$ as $\Delta h \to 0$ $\int_b^c$ & $\int_d^a$ become zero.

hence $\int_a^b \bar{E} \cdot d\bar{l} + \int_c^d \bar{E} \cdot d\bar{l} = 0$

a-b in dielectric 1 hence $\bar{E}$ is $E_{tan1}$

$$\therefore \int_a^b \bar{E} \cdot d\bar{l} = E_{tan1} \int_a^b d\bar{l} = E_{tan1} \Delta W$$

c-d in dielectric 2 hence $\bar{E}$ is $E_{tan2}$ c-d is opposite to a-b.

$$\therefore \int_c^d \bar{E} \cdot d\bar{l} = -E_{tan2} \Delta W$$

$$\therefore E_{tan1} \Delta W - E_{tan2} \Delta W = 0$$

$$\boxed{\therefore E_{tan1} = E_{tan2}}$$

thus the tangential components of field intensity at the boundary in both the dielectrics remain same i.e., $\bar{E}$ is continuous across the boundary

$$\bar{D} = \varepsilon \bar{E}$$

$$D_{tan1} = \varepsilon_1 E_{tan1} \quad \& \quad D_{tan2} = \varepsilon_2 E_{tan2}$$

$$\frac{D_{tan1}}{\varepsilon_1} = \frac{D_{tan2}}{\varepsilon_2}$$

$$\frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$

thus the tangential components of $\bar{D}$ undergoes some change across the interface hence tangential $\bar{D}$ is said to be discontinuous across the boundary.

to find normal components,

consider a Gaussian surface in the form of right circular cylinder, placed in such a way that half of it lies in dielectric-1 while the remaining half in dielectric 2.

(8)

$$\therefore \oint \bar{D}\cdot d\bar{S} = Q.$$

$$\int_{top} \bar{D}\cdot d\bar{S} + \int_{bottom} \bar{D}\cdot d\bar{S} + \int_{lateral\,(sp's)} \bar{D}\cdot d\bar{S} = Q.$$

$$\int_{lateral} \bar{D}\cdot d\bar{S} = 0 \quad \text{as } \Delta h \to 0$$

$$\therefore \int_{top} \bar{D}\cdot d\bar{S} + \int_{bottom} \bar{D}\cdot d\bar{S} = Q.$$

$$\therefore |\bar{D}| = D_{N1} \text{ for } 1$$

$$= D_{N2} \text{ for } 2$$

$$\therefore \int_{top} \bar{D}\cdot d\bar{S} = D_{N1} \int_{top} d\bar{S} = D_{N1}\Delta S$$

for top surface, the direction of $D_N$ is entering the boundary while for bottom surface, the direction of $D_N$ is leaving the boundary. Both are opposite in direction, at the boundary.

$$\therefore \int_{bottom} \bar{D}\cdot d\bar{S} = -D_{N2}\int_{bottom} d\bar{S} = -D_{N2}\Delta S$$

$$\therefore D_{N1}\Delta S - D_{N2}\Delta S = Q$$

$$Q = \rho_S \Delta S$$

$$\therefore D_{N1} - D_{N2} = \rho_S$$

There is no free charges available in perfect dielectric hence no free charge can exist on the surface. All charges are bound charges

$$\therefore \rho_S = 0.$$

$$D_{N1} = D_{N2}$$

hence the normal components is continuous at the boundary b/w the 2 perfect dielectrics.

$$D_{N1} = \varepsilon_1 E_{N1}$$

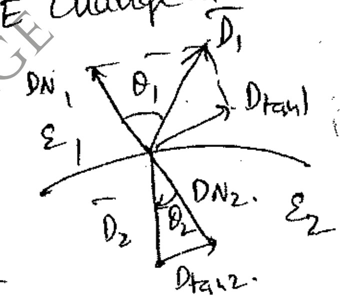
$$D_{N2} = \varepsilon_2 E_{N2}$$

$$\frac{D_{N1}}{D_{N2}} = \frac{\varepsilon_1}{\varepsilon_2} \frac{E_{N1}}{E_{N2}} = 1$$

$$\therefore \frac{E_{N1}}{E_{N2}} = \frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}}$$

The normal components of the $\bar{E}$ are inversely proportional to the relative permittivities of the two media.

Refraction of $\bar{D}$ at boundary: The direction of $\bar{D}$ & $\bar{E}$ change at the boundary b/w the two dielectrics. Let $\bar{D}_1$ & $\bar{E}_1$ make an angle $\theta_1$ with the normal to the surface. $\bar{D}_1$ & $\bar{E}_1$ direction is same as $\bar{D}_1 = \varepsilon_1 \bar{E}_1$.




Let $|\bar{D}_1| = D_1$ & $|\bar{D}_2| = D_2$

$$D_{N1} = D_1 \cos\theta_1 \rightarrow ②$$

$$D_{N2} = D_2 \cos\theta_2 \rightarrow ③$$

but $D_{N1} = D_{N2}$

$$\therefore D_1 \cos\theta_1 = D_2 \cos\theta_2$$

$$\frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2} \rightarrow ①$$

$$D_{tan1} = D_1 \sin\theta_1$$

$$D_{tan2} = D_2 \sin\theta_2 \rightarrow ④$$

$$\frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{D_{tan1}}{D_{tan2}}$$

$$\tan\theta_1 = \frac{D_{tan1}}{D_{N1}} ; \tan\theta_2 = \frac{D_{tan2}}{D_{N2}}$$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{D_{tan1}\, D_{N2}}{D_{tan2}\, D_{N1}}$$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2}$$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$

This is called law of refraction. Thus the angles $\theta_1$ & $\theta_2$ are dependent on the permittivities of two media and not on $\bar{D}$ & $\bar{E}$

Thus if $\varepsilon_1 > \varepsilon_2$, then $\theta_1 > \theta_2$

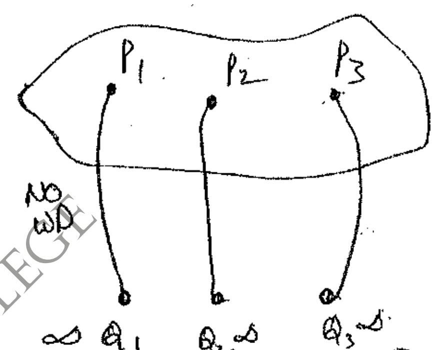
# ENERGY DENSITY IN THE ELECTROSTATIC FIELDS:-



Consider an empty space where there is no electric field at all. The charge $Q_1$ is moved from $\infty$ to point in the space $P_1$. This requires no work as there is no $\vec{E}$ present.

Now charge $Q_2$ is to be placed at point $P_2$ in space as shown in fig. but now there is an electric field due to $Q_1$ & $Q_2$ is required to be moved against the field of $Q_1$

hence the work is required to be done.




potential = WD/unit charge $\left(\frac{W}{Q}\right)$

$$WD = V \times Q$$

$\therefore$ WD to position $Q_2$ at $P_2 = V_{2,1} Q_2$ [$V_{2,1}$ = potential at $P_2$ due to $P_1$]

Charge $Q_3$ from $\infty$ to $P_3$ ; $Q_1$ & $Q_2$

$\therefore$ WD to position $Q_3$ at $P_3 = V_{3,1} Q_3 + V_{3,2} Q_3$

Thus for charge $Q_n$ at $P_n = V_{n,1} Q_n + V_{n,2} Q_n + \ldots$

hence the total WD in positioning all the charges is,

$$W_E = Q_2 V_{2,1} + Q_3 V_{3,1} + Q_3 V_{3,2} + \ldots \rightarrow (1)$$

If charges are placed in reverse order

$$W_E = Q_3 V_{3,4} + Q_2 V_{2,3} + Q_2 V_{2,4} + Q_1 V_{1,2} + Q_1 V_{1,3} + \ldots \rightarrow (2)$$

$Q_n$ is placed I then $Q_{n-1}$ finally $Q_1$

(1) + (2)

$$2W_E = Q_1 (V_{1,2} + V_{1,3} + V_{1,4} + \ldots + V_{1,n})$$
$$+ Q_2 (V_{2,1} + V_{2,3} + V_{2,4} + \quad + V_{2,n})$$
$$+ Q_3 (V_{3,1} + \quad + V_{3,n}) + \ldots$$

$\therefore V_{1,2} + V_{1,3} + \cdots + V_{1,n} = V_1$

$= V_2$

$$2W_E = Q_1 V_1 + Q_2 V_2 + Q_3 V_3 + \cdots$$

$$W_E = \frac{1}{2} \sum_{m=1}^{n} Q_m V_m \text{ Joule.}$$

This is the potential energy stored in the system of 'n' point c

$\rho_L \qquad W_E = \frac{1}{2} \int \rho_L \, dL \, V \; J$

$\rho_S \qquad = \frac{1}{2} \int \rho_S \, ds \, V \; J$

$\rho_V \qquad = \frac{1}{2} \int \rho_V \, dv \, V \; J$

ENERGY STORED INTERMS OF $\bar{D}$ & $\bar{E}$ :-

$$W_E = \frac{1}{2} \int_{vol} \rho_V \, V \, dv.$$

$$\rho_V = \nabla \cdot \bar{D} \quad \text{(Maxwell's I)}$$

$$W_E = \frac{1}{2} \int_{vol} (\nabla \cdot \bar{D}) V \, dv.$$

for any vector A & scalar V there is vector identity,

$$\nabla \cdot V\bar{A} = \bar{A} \cdot \nabla V + V(\nabla \cdot \bar{A})$$

$$(\nabla \cdot \bar{A}) V = \nabla \cdot V\bar{A} - \bar{A} \cdot \nabla V.$$

$$\therefore W_E = \frac{1}{2} \int_{vol} (\nabla \cdot V\bar{D} - \bar{D} \cdot \nabla V) dv$$

$$= \frac{1}{2} \int_{vol} (\nabla \cdot V\bar{D}) dv - \frac{1}{2} \int_{vol} \bar{D} \cdot \nabla V \, dv$$

according to divergence theorem, volume integral can be converted to closed surface integral if closed surface totally surrounds the volume. (10)

$$\frac{1}{2}\int_{vol} (\nabla\cdot V\bar{D})\,dv = \frac{1}{2}\oint (V\bar{D})\cdot d\bar{S}$$

$$W_E = \frac{1}{2}\oint (V\bar{D})\cdot d\bar{S} - \frac{1}{2}\int_{vol} \bar{D}\cdot\nabla V\,dV$$

$V = \frac{Q}{4\pi\epsilon_0 r}$

we know $V \propto \frac{1}{r}$ for dipoles $V \propto \frac{1}{r^2}$ $|\bar{D}| = \frac{Q}{4\pi r^2}$

$(\bar{D}) = \frac{Q}{4\pi r^2}$ point charge. $\bar{D} \propto \frac{1}{r^2}$ $\bar{D} \propto \frac{1}{r^3}$.

$\therefore V\bar{D} \propto \frac{1}{r^3}$ while ds varies as $r^2$. hence total integral varies as $1/r$. as surface becomes very large, $r\to\infty$ & $1/r \to 0$ hence closed surface integral is zero

$$W_E = -\frac{1}{2}\int_{vol} \bar{D}\cdot\nabla V\,dV$$

$$\bar{E} = -\nabla V$$

$$W_E = -\frac{1}{2}\int_{vol} \bar{D}\cdot(-\bar{E})\,dV$$

$$W_E = \frac{1}{2}\int_{vol} \bar{D}\cdot\bar{E}\,dV \text{ joules.}$$

$$\bar{D} = \epsilon_0\bar{E}$$

$$W_E = \frac{1}{2}\int_{vol} \epsilon_0 E^2\,dV \quad J$$

$$W_E = \frac{1}{2}\int_{vol} \frac{D^2}{\epsilon_0}\,dV \quad J.$$

In differential form,

$$dW_E = \frac{1}{2}\bar{D}\cdot\bar{E}\,dV$$

$$\boxed{\frac{dW_E}{dV} = \frac{1}{2}\bar{D}\cdot\bar{E} \quad J/m^3} \times V$$

energy density

if this / over the volume total energy

$$W_E = \int_{vol}\left(\frac{dW_E}{dV}\right)dV$$

$$\boxed{W_E = \frac{1}{2}\int_{V}\bar{D}\cdot\bar{E}\,dV} \; J$$

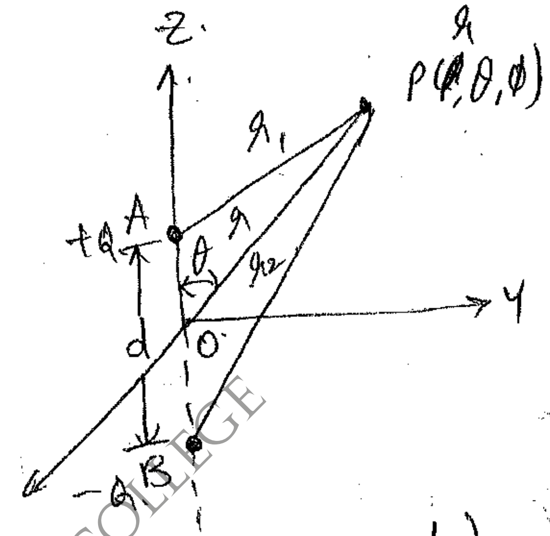
## ELECTRIC DIPOLE:-

The two point charges of equal magnitude but opposite sign, seperated by a very small distance gives rise to an electric dipole.

Consider an electric dipole.

$+Q$ & $-Q$, $d$.

$P(r, \theta, \phi)$ in spherical coordinate system.




$\bar{E}$ at $P = ?$

$P$ from $A = r_1$

$P$ from $B = r_2$

$P$ from $O = r$

the coordinates of $A$ are $(0, 0, d/2)$ & $B$ are $(0, 0, -d/2)$.

$$\bar{E} = -\nabla V$$

## EXPRESSION OF $\bar{E}$ DUE TO AN ELECTRIC DIPOLE:-

In spherical

the potential at point $P$ due to charge $+Q$

$$V_1 = \frac{+Q}{4\pi\varepsilon_0 r_1}$$

$$V_2 = \frac{-Q}{4\pi\varepsilon_0 r_2}$$

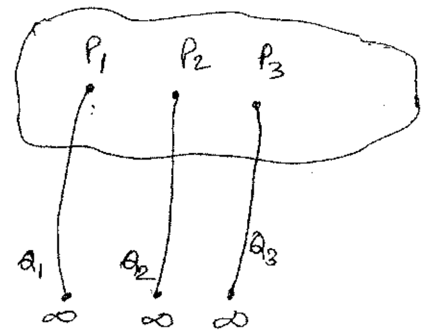
$$V = V_1 + V_2$$

$$= \frac{Q}{4\pi\varepsilon_0 r_1} - \frac{Q}{4\pi\varepsilon_0 r_2}$$

$$V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right]$$

# Energy Density in the Electrostatic Fields :

* To maintain a charge at a point in field




$$W = -Q\int \cancel{\vec{E}}^{\,0} \cdot d\vec{L} \Rightarrow \underline{W_1 = 0}$$

$$V = \frac{W}{Q} \Rightarrow \boxed{W = VQ}$$

$$W_2 = V_{2,1}Q_2$$

$$W_3 = V_{3,1}Q_3 + V_{3,2}Q_3 \;;\; W_4 = V_{4,1}Q_4 + V_{4,2}Q_4 + V_{4,3}Q_4.$$

Hence Total W.D $W_E = W_1 + W_2 + W_3 = \left(0 + V_{2,1}Q_2 + V_{3,1}Q_3 + V_{3,2}Q_3\right)$

Reverse order $W_E = 0 + V_{2,3}Q_2 + V_{1,3}Q_1 + V_{1,2}Q_1$

$$2W_E = (V_{1,3} + V_{1,2})Q_1 + (V_{2,3} + V_{2,1})Q_2 + (V_{3,1} + V_{3,2})Q_3.$$

$$W_E = \frac{1}{2}(V_1Q_1 + V_2Q_2 + V_3Q_3)$$

~~for~~ 'n' charges $W_E = \frac{1}{2}(V_1Q_1 + V_2Q_2 + \cdots + V_nQ_n)$

$$\boxed{W_E = \frac{1}{2}\sum_{i=1}^{n} V_iQ_i} \text{ Joules.}$$

Line $\rightarrow W_E = \frac{1}{2}\int \rho_L V dL$ Joules

surf $\rightarrow W_E = \frac{1}{2}\int \rho_s ds\, V$ Joules

Volume $\rightarrow W_E = \frac{1}{2}\int \rho_V dV\, V$ Joules

---

Energy Stored $\bar{D}$ & $\bar{E}$.

$$W_E = \frac{1}{2}\int_V \rho_V dV\, V$$

$$\rho_V = \nabla \cdot \bar{D}$$

$$W_E = \frac{1}{2}\int_V (\nabla \cdot \bar{D}) V\, dV$$

$$\nabla \cdot V\bar{A} = \bar{A} \cdot (\nabla V) + V(\nabla \cdot \bar{A})$$

$$W_E = \frac{1}{2}\int \left(\nabla \cdot V\bar{D} - \bar{D} \cdot (\nabla V)\right) dV$$

$$V = \frac{Q}{4\pi\epsilon_0 r} \;;\; |\bar{D}| = \frac{Q}{4\pi r^2}.$$

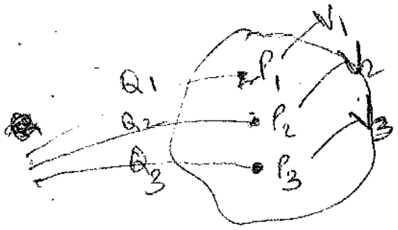
$$V\bar{D} \propto \frac{1}{r^3} \;;\; \text{dipole} \rightarrow V\bar{D} \propto \frac{1}{r^5}.$$

$$W_E = -\frac{1}{2}\int_V \bar{D} \cdot (\nabla V)\, dV = \boxed{\frac{1}{2}\int \bar{D} \cdot \bar{E}\, dV} \;\; \bar{E} = -\nabla V$$

Joules

$$\boxed{\frac{dW_E}{dV} = \frac{1}{2}\bar{D} \cdot \bar{E}} \;\; J/m^3$$

Energy density in electrostatic field :




$W_E = W_1 + W_2 + W_3$ $\qquad V = -\int \overline{E} \cdot d\overline{l}$

$W_E = 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32})$ $\qquad V = \frac{W}{Q} \Rightarrow W = QV$

$W_E = W_3 + W_2 + W_1$

$W_E = 0 + Q_2 (V_{23}) + Q_1 (V_{13} + V_{12})$

$\therefore \; 2W_E = Q_1 (V_{12} + V_{13}) + Q_2 (V_{21} + V_{23}) + Q_3 (V_{31} + V_{32})$

$W_E = \frac{1}{2} \left[ Q_1 V_1 + Q_2 V_2 + Q_3 V_3 \right]$

$W_E = \frac{1}{2} \sum_{k=1}^{N} Q_k V_k$ (or) $W_E = \frac{1}{2} \int_v \rho_v V \, dV$

$W_E = \frac{1}{2} \int (\nabla \cdot \overline{D}) V \, dV$

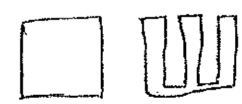
$(\nabla \cdot VA) = A \cdot \nabla V + V(\nabla \cdot A)$

$V(\nabla \cdot \overline{A}) = (\nabla \cdot V\overline{A}) - \overline{A} \cdot \nabla V$

$= \frac{1}{2} \int_v (\nabla \cdot V\overline{D}) \, dv - \frac{1}{2} \int (\overline{D} \cdot \nabla V) \, dV$

$= \frac{1}{2} \oint_s \cancel{V\overline{D} \cdot \overline{ds}}^{\,0} - \frac{1}{2} \int_v (\overline{D} \cdot \nabla V) \, dV$

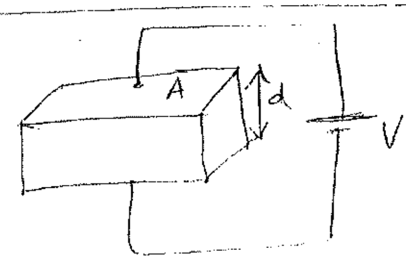
energy stored $\quad \boxed{W_E = +\frac{1}{2} \int (\overline{D} \cdot \overline{E}) \, dV}$ J



energy density $\quad \boxed{w_E = \frac{dW_E}{dV} = \frac{1}{2} \overline{D} \cdot \overline{E} = \frac{1}{2} \epsilon_0 E^2 = \frac{1}{2} \frac{D^2}{\epsilon_0}}$ $J/m^3$

---

Energy stored in a capacitor :




$E = \frac{V}{d}$

$W_E = \frac{1}{2} \int \overline{D} \cdot \overline{E} \, dV = \frac{1}{2} \epsilon_0 E^2 \int_v dv$

$= \frac{1}{2} \epsilon_0 \frac{V^2}{d^2} (Ad)$

$= \frac{1}{2} \left( \frac{\epsilon A}{d} \right) V^2$

$\boxed{W_E = \frac{1}{2} C V^2}$ J

Q. Det the 'C' of each of the capacitors in fig. Take $\epsilon_{r1} = 4$, $\epsilon_{r2} = 6$, $d = 5mm$, $S = 30cm^2$

CONCEPT OF CAPACITANCE:-

Consider two conducting materials $M_1$ and $M_2$ which are placed in a dielectric medium having permittivity $\varepsilon$.

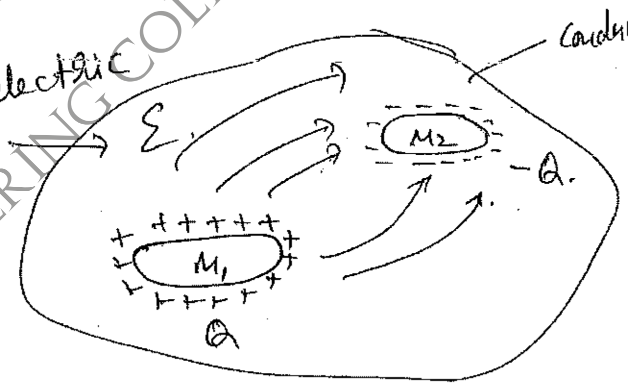
→ The material $M_1$ carries a $+^{ve}$ charge $Q$ } are equal
$M_2$ carries $-^{ve}$ $Q$.

→ There are no other charges present & total charge of the system is zero.

→ In conductor, charge can not reside within the conductor & it resides only on the surface.

→ such a system which has two conducting surfaces carrying equal and opposite charges, separated by a dielectric is called capacitive system giving rise to a capacitance.

→ The electric field is normal to the conductor surface and the electric flux is directed from $M_1$ towards $M_2$




→ There exist a pd b/w the 2 surfaces of $M_1$ & $M_2$.

The ratio of the magnitudes of the total charge on any of the two conductors & pd b/w the conductors is called the capacitance of the two conductor system.

$$C = \frac{Q}{V} \text{ farad}$$

$$\therefore V = -\int_L \bar{E} \cdot d\bar{L}$$

$$= -\int_{-}^{+} \bar{E} \cdot d\bar{L}$$

$$Q = \oint_S \bar{D} \cdot d\bar{S}$$

$$= \oint_S \varepsilon_0 \varepsilon_r \bar{E} \cdot d\bar{S}$$

$$= \oint_S \varepsilon \bar{E} \cdot d\bar{S}$$

$$\therefore C = \frac{Q}{V} = \frac{\oint_S \varepsilon \bar{E} \cdot d\bar{S}}{-\int_{-}^{+} \bar{E} \cdot d\bar{L}} \text{ Farad}$$

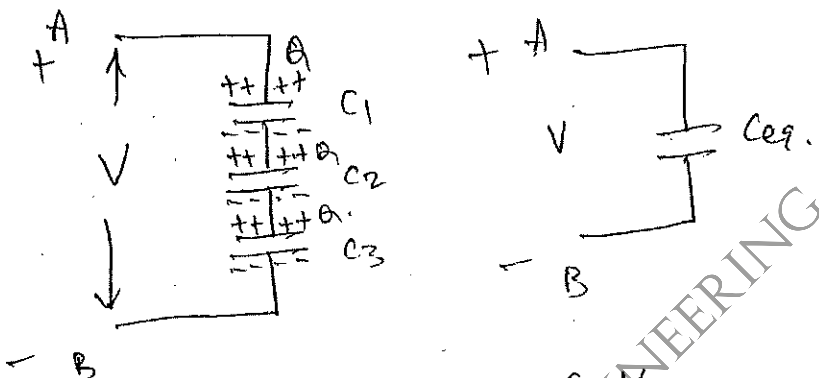
→ If charge $Q$ is increased, then $\bar{E}$ & $\bar{D}$ get increased by same factor.
the voltage $V$ also increased by same factor.

→ Thus the ratio $Q$ to $V$ remains constant as $C$.

→ Hence capacitance is not the function of charge, field intensity, flux density and potential difference.

→ It depends on physical dimensions of the system & the properties of the dielectric such as permittivity of the dielectric.

CAPACITOR IN SERIES:-




$$Q = C_1 V_1 = C_2 V_2 = C_3 V_3$$

$$V_1 = \frac{Q}{C_1} \quad V_2 = \frac{Q}{C_2} \quad V_3 = \frac{Q}{C_3}$$

$$C_{eq} = \frac{Q}{V} \text{ or } V = \frac{Q}{C_{eq}}$$

$$V = V_1 + V_2 + V_3$$

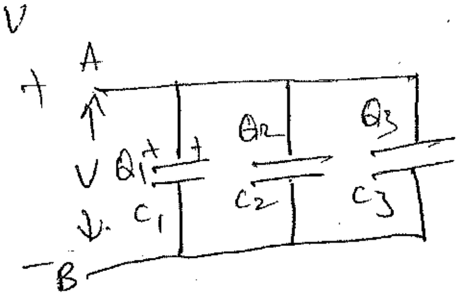
$$\frac{Q}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}$$

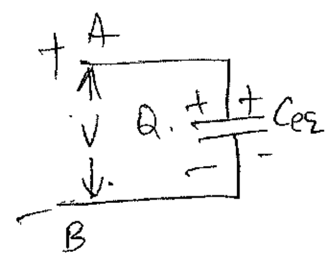
$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

⇒ for all the capacitors in series, the charge on all of them is always same, but voltage across them is different.

## CAPACITORS IN PARALLEL:-

$Q_1 = C_1 V, \; Q_2 = C_2 V, \; Q_3 = C_3 V$



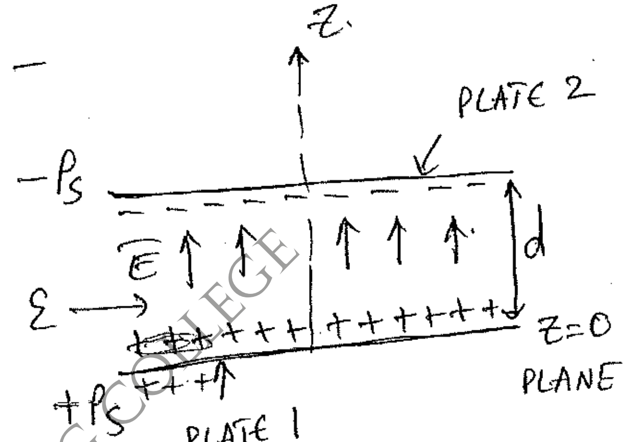

$Q = Q_1 + Q_2 + Q_3$

$= (C_1 + C_2 + C_3) V$

$Q = C_{eq} V$

$C_{eq} = C_1 + C_2 + C_3$

## PARALLEL PLATE CAPACITOR:-




→ It consists of two parallel metallic plates separated by distance 'd'.

→ The space b/w the plates is filled with a dielectric of permittivity $\varepsilon$.

→ The lower plate 1 carries +ve charge density $\rho_s$
upper plate 2 −ve $-\rho_s$

→ The plate 1 is placed in Z=0 i.e, XY plane hence normal to it is Z direction.

→ The plate 2 is in Z=d plane, parallel to XY plane.

Let A = Area of cross section of the plates in $m^2$

$$Q = \rho_s A \; C \rightarrow (1)$$

assuming plate 1 to be infinite sheet of charge

$$\bar{E}_1 = \frac{\rho_s}{2\varepsilon} \bar{a}_N$$

$$= \frac{\rho_s}{2\varepsilon} \bar{a}_z \; V/m$$

The $\bar{E}_1$ is normal at the boundary b/w conductor & dielectric without any tangential component

for plate-2 $\quad \bar{E}_2 = \frac{-\rho_s}{2\varepsilon}\bar{a}_N$

$$= \frac{-\rho_s}{2\varepsilon}(-\bar{a}_z)\ V/m \quad (\bar{E} \text{ is in upward direction})$$

$$\therefore \bar{E} = \bar{E}_1 + \bar{E}_2 = \frac{\rho_s}{2\varepsilon}\bar{a}_z + \frac{\rho_s}{2\varepsilon}\bar{a}_z$$

$$= \frac{\rho_s}{\varepsilon}\bar{a}_z$$

$$\therefore V = -\int_{-}^{+}\bar{E}\cdot d\bar{L} = -\int_{\text{upper}}^{\text{lower}} \frac{\rho_s}{\varepsilon}\bar{a}_z \cdot d\bar{L}$$

$$\therefore d\bar{L} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$$

$$V = -\int_{z=d}^{z=0} \frac{\rho_s}{\varepsilon}\bar{a}_z \cdot \left[dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z\right]$$

$$= -\int_{z=d}^{z=0} \frac{\rho_s}{\varepsilon}\,dz$$

$$= -\frac{\rho_s}{\varepsilon}\left[z\right]_d^0$$

$$= -\frac{\rho_s[-d]}{\varepsilon}$$

$$V = \frac{\rho_s d}{\varepsilon}\ V$$

$$C = \frac{Q}{V} = \frac{\rho_s A}{\frac{\rho_s d}{\varepsilon}} = \frac{\varepsilon A}{d}\ F$$

Thus if $\varepsilon = \varepsilon_0 \varepsilon_r$

$$\boxed{C = \frac{\varepsilon_0 \varepsilon_r A}{d}\ \text{Farad}}$$

capacitance depends on

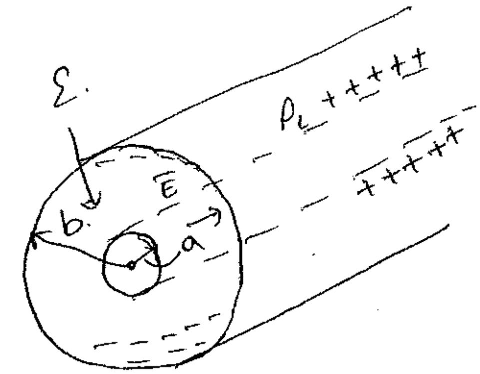
$\varepsilon_r$

$A$

$d$

## CAPACITANCE OF A CO-AXIAL CABLE:-

Consider a co-axial cable or co-axial capacitor as shown in fig.




Let a = Inner radius
b = outer radius

The two concentric conductors are separated by dielectric of permittivity ε.

The length of the cable is L m

The inner conductor carries a charge density $+\rho_L$ c/m on its surface then equal & opposite charge density $-\rho_L$ c/m exist on the outer conductor.

$$Q = \rho_L \times L \rightarrow (1)$$

assuming cylindrical coordinate system, $\bar{E}$ will be rad from inner to outer conductor, & for infinite line charge

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon r}\,\bar{a}_r$$

$\bar{E}$ is directed from inner to the outer conductor. The pd is WD in moving unit charge against $\bar{E}$ i.e., $r=b$ to $r=a$

$$V = -\int_{-}^{+} \bar{E}\cdot d\bar{l} = -\int_{r=b}^{r=a} \frac{\rho_L}{2\pi\varepsilon r}\,\bar{a}_r \cdot dr\,\bar{a}_r$$

$$= -\frac{\rho_L}{2\pi\varepsilon} \ln\left[\frac{a}{b}\right]$$

$$\therefore V = \frac{\rho_L}{2\pi\varepsilon} \ln\left[\frac{b}{a}\right]$$

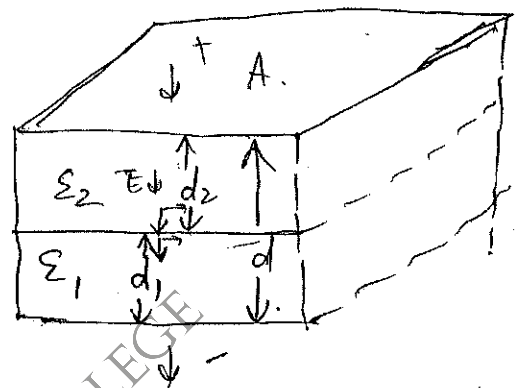
$$C = \frac{Q}{V} = \frac{\cancel{\rho_L} \times L}{\frac{\cancel{\rho_L}}{2\pi\varepsilon} \ln\left[\frac{b}{a}\right]} \Rightarrow C = \frac{2\pi\varepsilon L}{\ln\left[\frac{b}{a}\right]}\ F$$

## COMPOSITE PARALLEL PLATE CAPACITOR :-

A composite parallel plate capacitor is one in which the space b/w the plates is filled with more than one dielectric.

Consider a composite capacitor with space filled with 2 separate dielectrics for the distances $d_1$ & $d_2$

The dielectric interface is parallel to the conducting plates.




$d_1, \varepsilon_1$

$d_2, \varepsilon_2$

Let Q = charge on each plate

$\overline{E_1}$ = field intensity in region $d_1$

$\overline{E_2}$ = field " " " $d_2$

both the intensities are uniform

$$V_1 = E_1 d_1$$

$$V_2 = E_2 d_2$$

$$V = V_1 + V_2 = E_1 d_1 + E_2 d_2$$

at a dielectric - dielectric interface, $D_{N_1} = D_{N_2}$.

$$D_1 = \varepsilon_1 E_1$$

$$D_2 = \varepsilon_2 E_2$$

$$\therefore V = \frac{D_1}{\varepsilon_1} d_1 + \frac{D_2}{\varepsilon_2} d_2$$

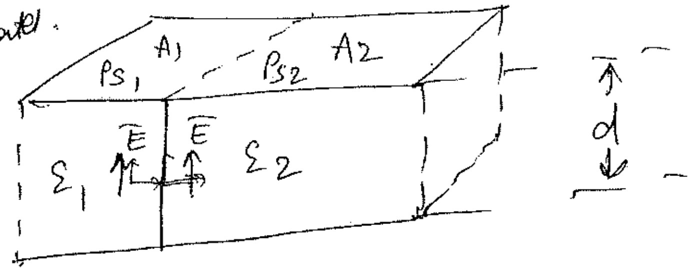
$$\boxed{D_1 = D_2 = \rho_S}$$

$$V = \rho_S \left[ \frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2} \right]$$

$$C = \frac{Q}{V} = \frac{Q}{\rho_S \left( \frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2} \right)}$$

## DIELECTRIC BOUNDARY NORMAL TO THE PLATES

- Consider the composite capacitor in which dielectric boundary is normal to the conducting plates.




→ The dielectric $\varepsilon_1$ occupying area A1 of the plates & the dielectric $\varepsilon_2$ occupying the area $A_2$ as shown in fig.

→ The total potential across the 2 plates is V & distance b/w the plates is d. Hence $\bar{E} = \frac{V}{d}$

→ At the boundary, both $\bar{E}_1$ & $\bar{E}_2$ are tangential & for dielectric-dielectric interface tangential components are equal

$$E_{tan1} = E_{tan2} = E_1 = E_2 = \frac{V}{d}$$

$$D_1 = \varepsilon_1 E_1 \qquad D_2 = \varepsilon_2 E_2$$

$$D_1 = \frac{\varepsilon_1 V}{d} \qquad D_2 = \frac{\varepsilon_2 V}{d}$$

→ on the plate the charge is divided into 2 parts.

on area $A_1$ , $\rho_{s_1} = D_1$

$A_2$ , $\rho_{s_2} = D_2$

$$Q = Q_1 + Q_2 = \rho_{s_1} A_1 + \rho_{s_2} A_2 = D_1 A_1 + D_2 A_2$$

$$Q = \frac{\varepsilon_1 V A_1}{d} + \frac{\varepsilon_2 V A_2}{d}$$

$$C = \frac{Q}{V} = \frac{\frac{\varepsilon_1 V A_1}{d} + \frac{\varepsilon_2 V A_2}{d}}{V}$$

$$C = \frac{\varepsilon_1 A_1}{d} + \frac{\varepsilon_2 A_2}{d} = C_1 + C_2$$

→ Thus if dielectric boundary is parallel to the plates, the arrangement is equivalent to 2 capacitors in series for which $C_{eq} = \frac{1}{\frac{1}{C_1}+\frac{1}{C_2}}$

→ if the dielectric boundary is normal to the plates, the arrangement is equivalent ... in parallel for which $C_{eq} = C_1 + C_2$

$$Q = \rho_s A.$$

$$C = \frac{\rho_s A}{\rho_s\left[\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}\right]}$$

$$C = \frac{A}{\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}}$$

$$C = \frac{1}{\frac{d_1}{\varepsilon_1 A} + \frac{d_2}{\varepsilon_2 A}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

$$C_1 = \frac{\varepsilon_1 A}{d_1} \quad \& \quad C_2 = \frac{\varepsilon_2 A}{d_2}$$

for a capacitor with 'n' dielectrics

$$C = \frac{A}{\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2} + \cdots + \frac{d_n}{\varepsilon_n}}.$$

* A parallel plate capacitor with a separation of 1 cm has 29 kV applied, when air was the dielectric used. Assume that the dielectric strength of air as 30 kV/cm. A thin piece of glass with $\varepsilon_r = 6.5$ with a dielectric strength of 290 kV/cm with thickness 0.2 cm is inserted. Find whether glass will break & air?

Sol:-
$$C_{air} = \frac{\varepsilon_0 A}{d_1} = \frac{8.854\times10^{-12} A}{0.8\times10^{-2}} = 1.1067\times10^{-9} A\ F$$

$$C_{glass} = \frac{\varepsilon_0 \varepsilon_r A}{d_2} = \frac{8.854\times10^{-12}\times 6.5 A}{0.2\times10^{-2}} = 2.8775\times10^{-8} A\ F$$

$C_{eq}$ = $C_{air}$ & $C_{glass}$ in series

$$= \frac{C_{air}\times C_{glass}}{C_{air} + C_{glass}} = 1.0657\times10^{-9} A\ F$$

let area of cross-section as $A = 1\,m^2$

$$C_{eq} = 1.06572\times10^{-9}\,F$$

$$Q = C_{eq}\times V = 3.0905\times10^{-5}\,C$$

$$V_{air} = \frac{Q}{C_{air}} = 27.9259\,kV$$

$$V_{glass} = V - V_{air} = 1.074\,kV$$

$$E_{air} = \frac{V_{air}}{d_1} = 3.49\times10^{6}\,V/m$$

$$= 34.9\,kV/cm.$$

$$E_{glass} = \frac{V_{glass}}{d_2} = \frac{1.074}{0.2} = 5.37\,kV/cm.$$

air will breakdown.

glass will not breakdown

* A parallel plate capacitor has a plate area of $1.5m^2$ & a plate seperation of 5mm. There are 2 dielectrics in b/w the plates. The I dielectric has a thickness of 3mm with a relative permittivity of 6 & second has a thickness of 2mm with a relative permittivity of 4. Find the capacitance.

Sol:-

$$C_1 = \frac{\varepsilon_1 A}{d_1} = \frac{\varepsilon_0 \varepsilon_{r_1} A}{d_1} = 26.562\,\mu F$$

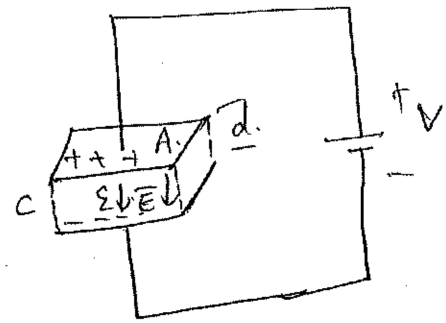
$$C_2 = \frac{\varepsilon_2 A}{d_2} = \frac{\varepsilon_0 \varepsilon_{r_2} A}{d_2} = 26.562\,\mu F$$

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2} = 13.281\,\mu F$$

(or)
$$C = \frac{A}{\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}} = \frac{1.5}{\frac{3\times10^{-3}}{\varepsilon_0\times6} + \frac{2\times10^{-3}}{\varepsilon_0\times4}} = 13.281\,\mu F.$$

## ENERGY STORED IN A CAPACITOR:-

Consider a parallel plate capacitor shown in fig.




$$\therefore \bar{E} = \frac{V}{d}\bar{a}_N$$

$$W_E = \frac{1}{2}\int_{vol} \bar{D}\cdot\bar{E}\, dv$$

$$= \frac{1}{2}\int_{vol} \varepsilon\bar{E}\cdot\bar{E}\, dv$$

$$= \frac{1}{2}\int_{vol} \varepsilon\,|\bar{E}|^2\, dv$$

$$= \frac{1}{2}\varepsilon\frac{V^2}{d^2}\int_{vol} dv$$

$$= \frac{1}{2}\varepsilon\frac{V^2 A d}{d^2}$$

$$= \frac{1}{2}\frac{\varepsilon A}{d}V^2$$

$$\therefore W_E = \frac{1}{2}CV^2 \text{ Joules.}$$

Energy density is the energy stored per unit volume as volume tends to zero

$$W_E = \frac{1}{2}C\int_{vol} |E|^2\, dv$$

$$W_E = \frac{1}{2}\varepsilon|\bar{E}|^2 \ J/m^3$$

$$\bar{D} = \varepsilon|\bar{E}|$$

$$W_E = \frac{1}{2}\frac{|\bar{D}|^2}{\varepsilon} = \frac{1}{2}|\bar{D}||\bar{E}| \ J/m^3.$$

* A parallel plate capacitor consists of two square metal plates with 500mm side and separated by 10mm. A slab of sulphur ($\epsilon_r = 4$) 6mm thick is placed on the lower plate & airgap of 4 mm. Find the capacitance of the capacitor.

$$C = 402.4545 \; PF$$

* In a cylindrical conductor of radius 2mm, the current density varies with the distance from the axis according to $J = 10^3 e^{-400r} \; A/m^2$. Find the total current.

Sol:

$$I = \int \bar{J} \cdot d\bar{s}$$

$$\bar{J} \cdot d\bar{s} = 10^3 e^{-400r} \, r \, dr \, d\phi$$

$$I = \int_{\phi=0}^{2\pi} \int_{r=0}^{0.002} 10^3 r e^{-400r} \, dr \, d\phi$$

$$= 10^3 (\phi)_0^{2\pi} \int_{r=0}^{0.002} r e^{-400r} \, dr$$

$$\int uv = u\int v - \int u' \int v$$

$$u = r, \; v = e^{-400r}$$

$$= 1.1955 \times 10^{-6}$$

$$\therefore I = 10^3 \times 2\pi \times 1.1955 \times 10^{-6} = 7.5115 \; mA.$$

① A parallel plate capacitor has a plate area of $1.5 m^2$ & a plate separation of 5mm. There are 2 dielectrics in b/w the plates. The I has a thickness of 3mm with $\varepsilon$ 6 & 2 of 2mm with 4. Find C.

$C_1 = 26.562 nF$

$C_2 = 26.562 nF$

$C_{eq} = 13.281 nF$

② A parallel plate capacitor consists of 2 square metal plates with 500 side & separated by 10mm. A slap of sulphur ($\varepsilon_r = 4$) 6mm thick is placed on the lower plate and air gap of 4mm. Find the capacitance of the capacitor.

Sol: $d_1 = 6\times10^{-3} m$ $\varepsilon_{r_1} = 4$

$d_2 = 4\times10^{-3} m$, $\varepsilon_{r_2} = 1$

$A = 500\times500 = 0.25 m^2$

$$C = \frac{1}{\frac{d_1}{\varepsilon_1 A} + \frac{d_2}{\varepsilon_2 A}} = 402.4545 pF$$

③ A parallel plate capacitor with air as dielectric has a plate area of $36\pi$ and a separation b/w the plates of 1mm. It is charged to 100V by connecting it across a battery. If the battery is disconnected and plate separation is increased to 2mm, calculate the change in ⓐ Pd across the plates & ⓑ energy stored.

Sol: $A = 36\pi\times10^{-4} m^2$, $V_1 = 100V$, $d_1 = 1\times10^{-3} m$ $\varepsilon_r = 1$

$$C_1 = \frac{\varepsilon_0 \varepsilon_r A}{d_1} = 100.1363 pF$$

$E_1 = \frac{1}{2} C_1 V_1^2 = 0.5 \mu J$

$Q = C_1 V_1 = 1\times10^{-8} C.$

battery disconnected

Charge Q remain same

$d_2 = 2\times10^{-3} m$

$C = C_2$

$V = V_2$

$Q = C_2 V_2$ & $C_2 = \frac{\varepsilon_0 \varepsilon_r A}{d_2} = 50.068 pF$

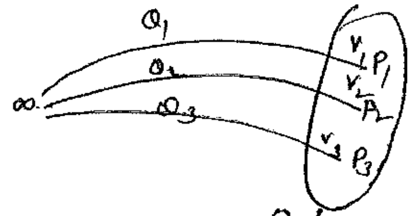
$1\times10^{-8} = 50.068\times10^{-12} V_2$

$V_2 = 199.72 V.$

$E_2 = \frac{1}{2} C_2 V_2^2 = 0.998 \mu J$

MA

# Energy density in electrostatic fields




$$W_E \Rightarrow Q_1 \cancel{V_1} + Q_2 \cdot (V_{21}) + Q_3 (V_{32} + V_{31})$$

$$\Rightarrow Q_3 (0) + Q_2 (V_{23}) + Q_1 (V_{12} + V_{13})$$

$$2 W_E \Rightarrow Q_1 (V_{12} + V_{13}) + Q_3 (V_{32} + V_{31}) + Q_2 (V_{21} + V_{23})$$

$$W_E \Rightarrow \frac{1}{2} Q_1 V_1 + Q_2 V_2 + Q_3 V_3$$

$$\boxed{W_E \Rightarrow \frac{1}{2} \sum_{k=1}^{n} Q_k V_k}$$

$$W_E = \frac{1}{2} \int_V (\nabla \cdot D) V \, dv \qquad \nabla \cdot D \Rightarrow \rho_v$$

$$W_E = \frac{1}{2} \int_V (\nabla \cdot D) V \cdot dv$$

$$\nabla \cdot VA \Rightarrow A \cdot \nabla V + V (\nabla \cdot A)$$

$$A.$$

$$\nabla \cdot VA - A \cdot \nabla V = V (\nabla \cdot A)$$

$$(\nabla \cdot D) V \Rightarrow \frac{1}{2} \nabla \cdot VD - D \cdot \nabla V$$

$$D \propto \frac{1}{r^2}$$

$$(\nabla \cdot D) V = - \varepsilon E (\nabla V)$$

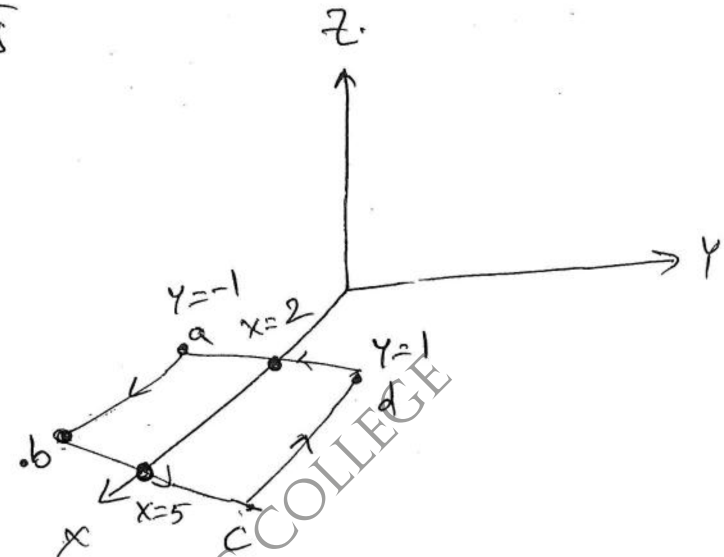
$$= \frac{1}{2} \varepsilon (\nabla V)^2$$

$$\nabla \cdot VA \Rightarrow A \cdot \nabla V - V(\nabla \cdot A)$$

$$\boxed{W_E \Rightarrow \frac{1}{2} \int_V \varepsilon |E|^2 dv}$$

3) Evaluate both sides of the Stoke's theorem for the field $\bar{H} = 6xy\,\bar{a_x} - 3y^2\,\bar{a_y}$ A/m and the rectangular path around the region, $2 \le x \le 5$, $-1 \le y \le 1$, $z = 0$. Let the positive direction of $d\bar{S}$ be $\bar{a_z}$.

Sol:
$$\oint_L \bar{H}\cdot d\bar{L} = \int_S (\nabla \times \bar{H})\, d\bar{S}$$




LHS:

$$\oint \bar{H}\cdot d\bar{L} = \int_{ab} + \int_{bc} + \int_{cd} + \int_{da} \bar{H}\cdot d\bar{L}$$

$$\int_{ab} \bar{H}\cdot d\bar{L} = \int_{x=2}^{5} (6xy\,\bar{a_x} - 3y^2\,\bar{a_y})\, dx\,\bar{a_x}$$

$= 63y$

for path ab, $y = -1 = -63$

$$\int_{bc} \bar{H}\cdot d\bar{L} = \int_{y=-1}^{1} -3y^2\, dy = -2$$

$$\int_{cd} \bar{H}\cdot d\bar{L} = \int_{x=5}^{2} 6xy\, dx = -63y \quad (\text{for path cd, } y = 1 \Rightarrow -63)$$

$= -63$

$$\int_{da} \bar{H}\cdot d\bar{L} = \int_{y=1}^{-1} -3y^2\, dy = 2$$

$\therefore \oint \bar{H}\cdot d\bar{L} = -63 - 2 - 63 + 2 = \underline{-126\,A.}$

RHS:

$$\nabla \times \bar{H} = \begin{vmatrix} \bar{a_x} & \bar{a_y} & \bar{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy & -3y^2 & 0 \end{vmatrix} = \bar{a_x}(0-0) + \bar{a_y}(0-0) + \bar{a_z}(0-6x) = -6x\,\bar{a_z}$$

$$\int_S (\nabla \times \bar{H})\cdot d\bar{S} = \int_S (-6x\,\bar{a_z})\cdot(dx\,dy\,\bar{a_z}) = \int_{y=-1}^{1}\int_{x=2}^{5} -6x\, dx\, dy$$

$d\bar{S} = dx\,dy\,\bar{a_z}$

$= \underline{-126\,A.}$

(1) In the region $0 < \rho < 0.5m$ in cylindrical coordinates, the current density is $\bar{J} = 4.5 e^{-2\rho} \bar{a_z}$ A/$m^2$ and $\bar{J} = 0$ elsewhere. Use ampere's circuital law to find $\bar{H}$.

Sol:
$$I = \oint \bar{J} \cdot \bar{ds}$$
$$\bar{ds} = \rho\, d\rho\, d\phi\, \bar{a_z} \quad (\because \bar{J} \text{ is in } \bar{a_z})$$
$$I = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{0.5} 4.5 e^{-2\rho} \bar{a_z} \cdot \rho\, d\rho\, d\phi\, \bar{a_z}$$
$$I = 1.8676 A.$$

Consider a closed path $\rho \geq 0.5m$ that encloses $I = 1.8676 A.$

$$\oint \bar{H} \cdot \bar{dL} = I$$
$$\bar{H} = H_\phi \bar{a_\phi}$$
$$\bar{dL} = \rho\, d\phi\, \bar{a_\phi}$$

$$\int_{\phi=0}^{2\pi} H_\phi \rho\, d\phi = I$$
$$H_\phi = \frac{1.8676}{2\pi\rho}$$
$$\bar{H} = \frac{0.2972}{\rho} \bar{a_\phi} \quad \text{A/m for } \rho \geq 0.5$$

(2) $\bar{H}$ due to a current source is given by $\bar{H} = [y\cos(\alpha x)]\bar{a_x} + (y + e^x)\bar{a_z}$. Describe current density over the yz plane.

Sol:
$$\nabla \times \bar{H} = \bar{J} \Rightarrow \begin{vmatrix} \bar{a_x} & \bar{a_y} & \bar{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\cos(\alpha x) & 0 & y + e^x \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(y + e^x)\right)\bar{a_x} + \left[\frac{\partial y\cos(\alpha x)}{\partial z} - \frac{\partial(y + e^x)}{\partial x}\right]\bar{a_y} + \left[-\frac{\partial}{\partial y} y\cos(\alpha x)\right]\bar{a_z}$$

$$= \bar{a_x} + (0 - e^x)\bar{a_y} + (-\cos\alpha x)\bar{a_z} \Rightarrow \text{on yz plane } x = 0$$

$$\bar{J} = \bar{a_x} - e^0 \bar{a_y} - \cos 0\, \bar{a_z} = \bar{a_x} - \bar{a_y} - \bar{a_z} \text{ A/m}^2$$

# UNIT - III
# MAGNETO STATICS

## TOSTATICS:-

## UNIT-III

poles N & S

The region around a magnet within which the influence of the magnet can be experienced is called Magnetic field. Field is represented by imaginary lines around the magnet which are called magnetic lines of force.

, electric flux lines originate from an isolated $+^{ve}$ charge & diverge to terminate at infinity; while a $-$ve charge, starting from infinity converge on a charge.

, But magnetic flux, the poles exist in pairs only an isolated magnetic pole can not exist.

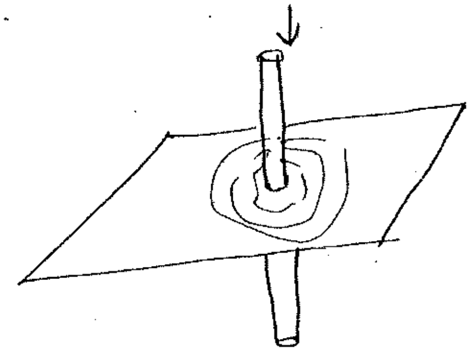
→ Magnetic flux lines exist in the form of closed loop.

→ Magnetic field due to current carrying conductor

→ Right hand thumb rule.

→ 

into the plane    out of the plane



Right handed screw rule.

## MAGNETIC FIELD INTENSITY:-

The magnetic field intensity at any point in the magnetic field is defined as the force experienced by a unit north pole of one wb strength, when placed at that point. denoted as $\bar{H}$.

A/m or N/wb or AT/m

$\bar{B} = \frac{F}{pole}$   $\bar{E} = \frac{F}{Q}$

MAGNETIC FLUX DENSITY:-

The total magnetic lines of force i.e., magnetic flux a unit area in a plane at right angles to the direction flux is called magnetic flux density, $\bar{B}$, vector, $wb/m^2$ or Tesla

RELATION B/W $\bar{B}$ and $\bar{H}$:-

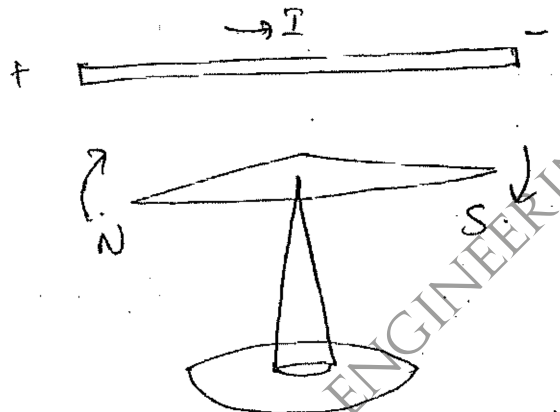
$\bar{B} = \mu \bar{H}$ property of the medium permeability

$\bar{B} = \mu_0 \mu_r \bar{H}$ $\mu_0 = 4\pi \times 10^{-7}$ for free space => $\mu_r = 1$

$\mu_r = 1$ for all non-magnetic media

OERSTED'S EXPERIMENT:-

$\mu_r > 1$ for magnetic materials.




A current carrying conductor is taken. A compass needle is kept under this conductor as shown.

When there is no current through the conductor, then needle is pointing along N & S of the earth.

But when the conductor carries current then the needle neither attracted to the conductor nor repelled from it but it moved and tended to stand at right angles to the conductor

⇒ from this experiment, oersted showed that an electric current produces a magnetic field.

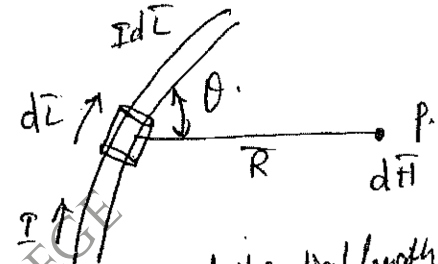
## BIOT SAVART LAW:-

Consider a conductor carrying a direct current $I$ and a steady magnetic field produced around it.

Consider a differential length $dL$ hence the different current element is $IdL$. The point $P$ is at a distance $R$ from the different current element.

Biot Savart law states that,

the magnetic field intensity $d\bar{H}$ produced at a point $P$ due to a differential current element $IdL$ is




1) proportional to the product of the current $I$ & differential length $dL$
2) The sine of the angle b/w the element & the line joining point $P$ to the element
3) and inversely proportional to the square of the distance $R$ b/w point $P$ & the element.

$$d\bar{H} \propto \frac{IdL\sin\theta}{R^2}$$

$$d\bar{H} = \frac{kIdL\sin\theta}{R^2}$$

$$k = \frac{1}{4\pi}$$

$$d\bar{H} = \frac{IdL\sin\theta}{4\pi R^2}$$

In vector form

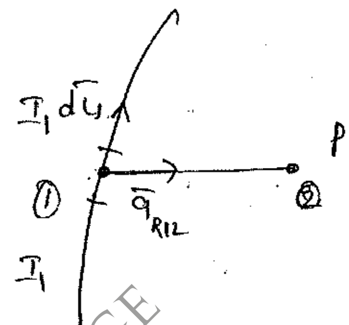
$dL$ = Magnitude of vector length $d\bar{L}$

$\bar{a}_R$ = unit vector in the direction from differential current element to point $P$

$$d\bar{L} \times \bar{a}_R = dL|\bar{a}_R|\sin\theta$$
$$= dL\sin\theta$$

$$d\bar{H} = \frac{I\, d\bar{L} \times \bar{a}_R}{4\pi R^2} \text{ A/m}$$

$$\therefore\ d\bar{H} = \frac{I\, d\bar{L} \times \bar{R}}{4\pi R^3}$$




$$\bar{H} = \oint \frac{I\, d\bar{L} \times \bar{a}_R}{4\pi R^2}$$

$$d\bar{H}_2 = \frac{I_1 \cdot d\bar{L}_1 \times \bar{a}_{R12}}{4\pi R_{12}^2} \text{ A/m}$$

$I_1$ = current flowing through $dL_1$ at point 1

$d\bar{L}_1$ = differential vector length at point 1

$\bar{a}_{R12}$ = unit vector in the direction from element at point 1 to the point P at point 2

$$\bar{a}_{R12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|}$$

$$\therefore\ \bar{H}_2 = \oint \frac{I_1\, d\bar{L}_1 \times \bar{a}_{R12}}{4\pi R_{12}^2} \text{ A/m}$$

This is called integral form of Biot-Savart law.

## BIOT-SAVART LAW INTERMS OF DISTRIBUTED SOURCES:-

(3) (3)

Consider a surface carrying a uniform current over its surface then the surface current density is denoted as $\bar{K}$ & measured in A/m.

Thus for uniform current density, the current $I$ in any width b is $\perp^r$ to the direction of current flow.

Thus if ds is the differential surface area considered of a surface having current density $\bar{K}$ then,

current element - A-m.

$$\boxed{I\,d\bar{L}} = \underset{A/m^2}{\bar{K}}\,ds$$

If the current density in a volume of a given conductor is $\bar{J}$ measured in A/m² then for a differential volume dv

$$I\,d\bar{L} = \underset{A/m^2}{\bar{J}}\,dv$$

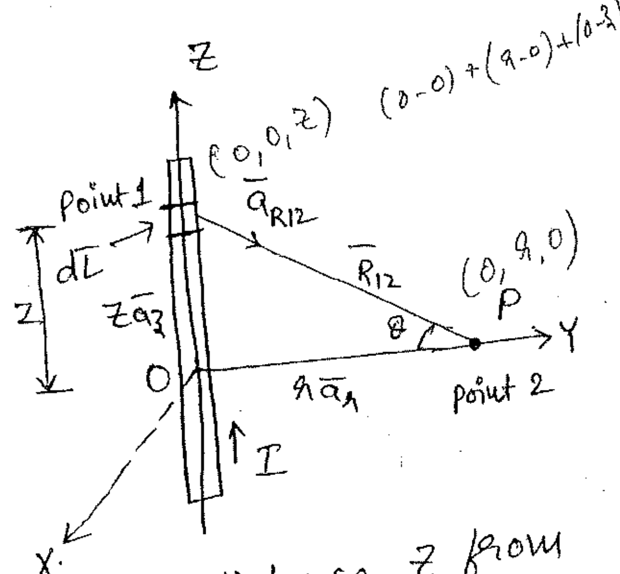
$$\bar{H} = \int_S \frac{\bar{K}\times\bar{a}_R\,ds}{4\pi R^2}\ A/m$$

$$\bar{H} = \int_{vol} \frac{\bar{J}\times\bar{a}_R\,dv}{4\pi R^2}\ A/m$$

## $\bar{H}$ DUE TO INFINITELY LONG STRAIGHT CONDUCTOR:-

Consider an infinitely long straight conductor, along Z-axis. The current passing through the conductor is a direct current of I A.

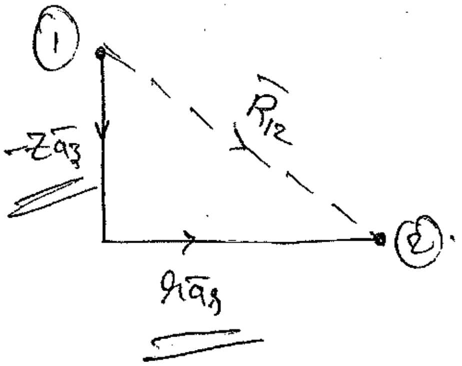
$\bar{H}$ at a point P is to be calculated, which is at a distance 'r' from the Z-axis.




Consider small differential element at point 1, along the Z-axis, at a distance z from origin.

$$\therefore\ I\,d\bar{L} = I\,dz\,\bar{a}_z$$

The distance vector joining point 1 to point 2 is $\overline{R_{12}}$




$$\overline{R_{12}} = -z\bar{a}_z + r\bar{a}_r$$

$$\bar{a}_{R12} = \frac{r\bar{a}_r - z\bar{a}_z}{\sqrt{r^2+z^2}}$$

$d\bar{l} = d\rho\, a_\rho + \rho\, d\phi\, a_\phi + dz\, a_z$

$$\therefore\ d\bar{L} \times \bar{a}_{R12} = \begin{vmatrix} \bar{a}_r & \bar{a}_\phi & \bar{a}_z \\ 0 & 0 & dz \\ r & 0 & -z \end{vmatrix} = r\, dz\, \bar{a}_\phi$$

$$\therefore\ I\, d\bar{L} \times \bar{a}_{R12} = \frac{I\, r\, dz\, \bar{a}_\phi}{\sqrt{r^2+z^2}}$$

$d\bar{L} = 0\bar{a}_x + 0\bar{a}_y + dz\,\bar{a}_z$

$\bar{a}_{R12} = r\bar{a}_r + 0\bar{a}_\phi - z\bar{a}_z$

$d\bar{L} \times \bar{a}_{R12} = \begin{vmatrix} \bar{a}_r & \bar{a}_\phi & \bar{a}_z \\ 0 & 0 & dz \\ r & 0 & -z \end{vmatrix} \Big/ \sqrt{r^2+z^2}$

$= 0\bar{a}_r - (r\, dz)\bar{a}_\phi + (0)\bar{a}_z$

$= \dfrac{I\, r\, dz\, \bar{a}_\phi}{\sqrt{r^2+z^2}^{\,2}}$

according to biot-savart law, $d\bar{H}$ at point 2 is,

$$d\bar{H} = \frac{I\, d\bar{L} \times \bar{a}_{R12}}{4\pi R_{12}^2} = \frac{I\, r\, dz\, \bar{a}_\phi}{4\pi\sqrt{r^2+z^2}\left(\sqrt{r^2+z^2}\right)^2}$$

$$= \frac{I\, r\, dz\, \bar{a}_\phi}{4\pi\left(r^2+z^2\right)^{3/2}}$$

over the entire length

$$\bar{H} = \int_{-\infty}^{\infty} \frac{I\, r\, dz\, \bar{a}_\phi}{4\pi\left(r^2+z^2\right)^{3/2}}$$

put $z = r\tan\theta$, $z^2 = r^2\tan^2\theta$

$dz = r\sec^2\theta\, d\theta$, $z = -\infty$, $\theta = -\pi/2$ & $z = \infty \Rightarrow \theta = \pi/2$

$$\bar{H} = \int_{-\pi/2}^{\pi/2} \frac{I\, r\, r\sec^2\theta\, d\theta\, \bar{a}_\phi}{4\pi\left(r^2 + r^2\tan^2\theta\right)^{3/2}} \qquad \left(1+\tan^2\theta = \sec^2\theta\right).$$

$$\boxed{\therefore\ \bar{H} = \frac{I}{2\pi r}\bar{a}_\phi\ \ A/m} \qquad \boxed{\bar{B} = \mu\bar{H} = \frac{\mu I}{2\pi r}\bar{a}_\phi\ \ wb/m^2}$$

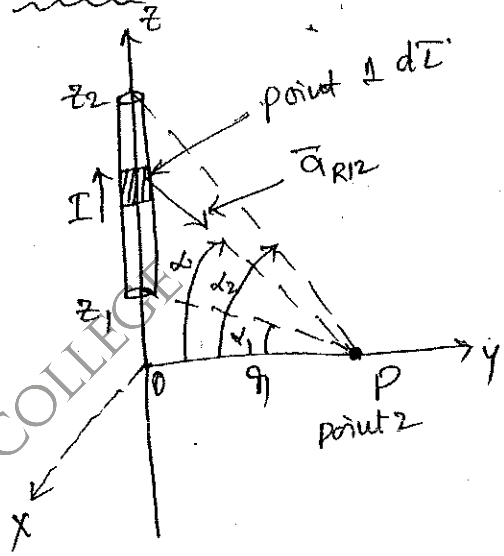
NOTE:-

1) The magnitude of $\bar{H}$ is not a function of $\phi$ & $z$.

- It is inversely proportional to '$r$' which is $\perp^r$ distance of the point from the conductor.

2) direction of $\bar{H}$ is tangential i.e along $\bar{a}_\phi$ which is into the plane of the paper at point P

## 2) $\bar{H}$ DUE TO STRAIGHT CONDUCTOR OF FINITE LENGTH:-




consider a conductor of finite length placed along z-axis,

- current I, $\perp^r$ distance of P from z is $r$

- $z = z_1$, $z = z_2$

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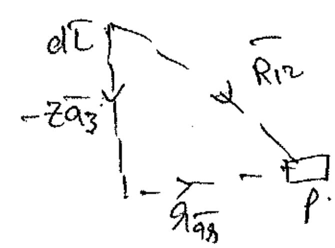
$\therefore d\bar{L} = dz\,\bar{a}_z$

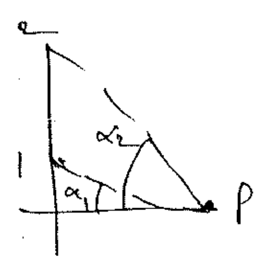
$$\therefore \bar{a}_{R12} = \frac{r\bar{a}_r - z\bar{a}_z}{\sqrt{r^2+z^2}}$$




$$\therefore I d\bar{L} \times \bar{a}_{R12} = \frac{I r\, dz\, \bar{a}_\phi}{\sqrt{r^2+z^2}}$$

$$d\bar{H} = \frac{I r\, dz\, \bar{a}_\phi}{4\pi (r^2+z^2)^{3/2}}$$

The direction is always normal to the plane containing the source and to be decided by right handed screw rule




$$\bar{H} = \int_{z_1}^{z_2} \frac{I r\, dz\, \bar{a}_\phi}{4\pi (r^2+z^2)^{3/2}}$$

$z = r\tan\alpha$, $z^2 = r^2\tan^2\alpha$,

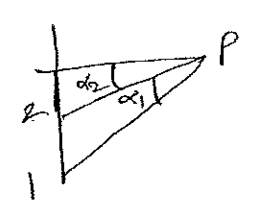
$dz = r\sec^2\alpha\, d\alpha$

for $z = z_1$, $z_1 = r\tan\alpha_1$

$z = z_2$ $\quad z_2 = r\tan\alpha_2$

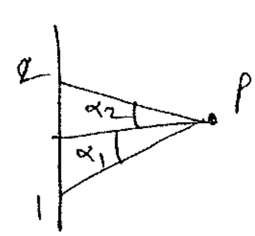
$\alpha_1 = \tan^{-1}\left(\frac{z_1}{r}\right)$

$\alpha_2 = \tan^{-1}\left(\frac{z_2}{r}\right)$



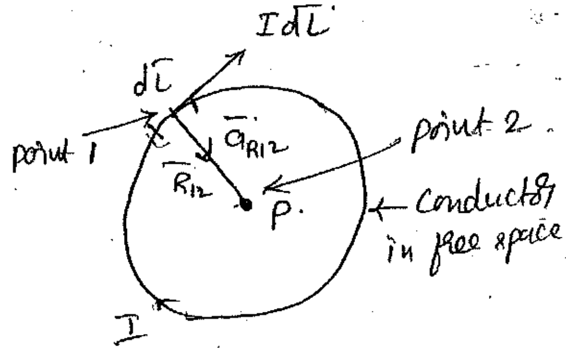

$$\bar{H} = \frac{I}{4\pi r}\left[\sin\alpha_2 - \sin\alpha_1\right]\bar{a}_\phi \quad A/m$$

$$\bar{B} = \frac{\mu I}{4\pi r}\left[\sin\alpha_2 - \sin\alpha_1\right]\bar{a}_\phi \quad Wb/m^2$$



## $\bar{H}$ AT THE CENTRE OF A CIRCULAR CONDUCTOR:-

Consider the current carrying conductor arranged in a circular form as shown in fig.




$\bar{H}$ at the centre

$d\bar{L}$ at point 1

Let $\theta$ = angle b/w $Id\bar{L}$ & $\bar{a}_{R12}$

$\bar{a}_{R12}$ -

$\bar{R}_{12}$ -

$$\therefore \; Id\bar{L} \times \bar{a}_{R12} = I\,|d\bar{L}|\,|\bar{a}_{R12}| \sin\theta\, \bar{a}_N$$

$$= I\,dL \sin\theta\, \bar{a}_N$$

$$d\bar{H} = \frac{Id\bar{L} \times \bar{a}_{R12}}{4\pi R_{12}^2} = \frac{I\,dL \sin\theta\, \bar{a}_N}{4\pi R^2}$$

$$\bar{H} = \oint \frac{I\,dL \sin\theta\, \bar{a}_N}{4\pi R^2}$$

$$= \frac{I \sin\theta\, \bar{a}_N}{4\pi R^2} \oint dL$$

$$= \frac{I \sin\theta\, \bar{a}_N}{4\pi R^2} \times 2\pi R$$

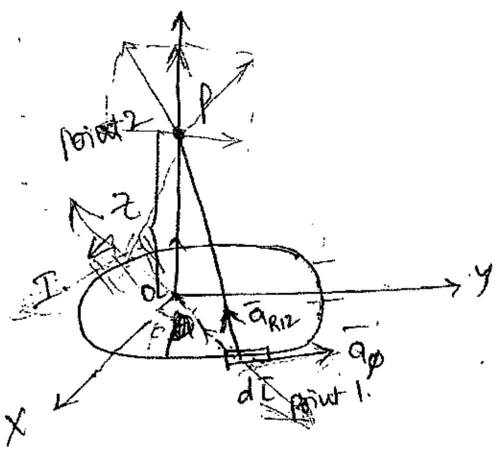
$$= \frac{I \sin\theta}{2R}\, \bar{a}_N$$

as $Id\bar{L}$ is tangential to the circle & $R_{12}$ is the radius, $\theta$ must be 90

$$\bar{H} = \frac{I \sin 90^\circ}{2R}\, \bar{a}_N = \frac{I}{2R}\, \bar{a}_z$$ ($\because$ $\bar{a}_N = \bar{a}_z$ if the circular loop is placed in xy plane)

$$\bar{B} = \frac{\mu_0 I}{2R}\, \bar{a}_N \; \text{wb/m}^2 \text{ free space.}$$

## $\bar{H}$ ON THE AXIS OF A CIRCULAR LOOP:-




$$d\bar{l} = d\rho\,\bar{a}_\rho + \rho\, d\phi\, \bar{a}_\phi + dz\, \bar{a}_z$$

z=0 plane.

$$I\,d\bar{l} = I\rho\, d\phi\, \bar{a}_\phi$$

$$\bar{R}_{12} = -\rho\bar{a}_\rho + z\bar{a}_z$$

$$d\bar{l} \times \bar{a}_{R12} = \begin{vmatrix} \bar{a}_\rho & \bar{a}_\phi & \bar{a}_z \\ 0 & \rho d\phi & 0 \\ -\rho & 0 & z \end{vmatrix} = z\rho d\phi\, \bar{a}_\rho + \rho^2 d\phi\, \bar{a}_z$$

$$\begin{array}{cccc} & \rho & \phi & z \\ 2 & 0 & 0 & z \\ 1 & \rho & 0 & 0 \end{array}$$

$(0-\phi)\bar{a}_\rho + (0-0)\bar{a}_\phi + (z-0)\bar{a}_z$

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$$d\bar{H} = \frac{I\, d\bar{l} \times \bar{a}_{R12}}{4\pi R_{12}^2}$$

$$= \frac{I\left[z\rho d\phi\,\bar{a}_\rho + \rho^2 d\phi\,\bar{a}_z\right]}{4\pi\sqrt{\rho^2+z^2}\left(\sqrt{\rho^2+z^2}\right)^2}$$

$$\bar{H} = \int_0^{2\pi} \frac{I\left(z\rho\bar{a}_\rho + \rho^2\bar{a}_z\right)d\phi}{4\pi\left(\rho^2+z^2\right)^{3/2}}$$

$$= \frac{I}{4\pi}\left[\int_0^{2\pi} \frac{z\rho d\phi}{\left(\rho^2+z^2\right)^{3/2}}\,\bar{a}_\rho + \int_0^{2\pi} \frac{\rho^2 \bar{a}_z}{\left(\rho^2+z^2\right)^{3/2}}\,d\phi\right]$$

$$\int_0^{2\pi} \frac{z\rho\, d\phi}{\left(\rho^2+z^2\right)^{3/2}}\left(\cos\phi\,\bar{a}_x + \sin\phi\,\bar{a}_y\right)$$

$$\Rightarrow \int_0^{2\pi} \cos\phi\, d\phi = 0.$$

$$= 0.$$

$$\therefore = 0$$

$$\bar{H} = \frac{I}{4\pi}\int_0^{2\pi} \frac{\rho^2 d\phi}{\left(\rho^2+z^2\right)^{3/2}}\,\bar{a}_z$$

$$= \frac{I\rho^2\bar{a}_z}{4\pi\left(\rho^2+z^2\right)^{3/2}}\int_0^{2\pi} d\phi$$

$$\bar{H} = \frac{I\rho^2}{2\left(\rho^2+z^2\right)^{3/2}}\,\bar{a}_z$$

when p is shifted to centre of the loop

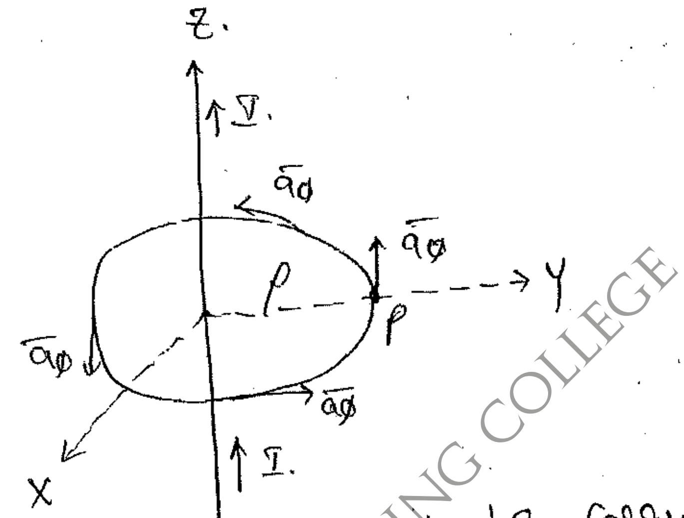
Radial Component
$\bar{H}$ at p cannot have any p...

$z=0$. $\quad \bar{H} = \frac{I\rho^2}{2\rho^{2\times 3/2}} = \frac{I}{2\rho}\bar{a}_z$ A/m

## AMPERE'S CIRCUITAL LAW:-

It states that, the line integral of $\overline{H}$ around a closed path is exactly equal to the direct current enclosed by that path.

$$\boxed{\oint \overline{H} \cdot d\overline{l} = I}$$




Consider a long straight conductor carrying direct current I placed along z axis as shown in fig. Consider a closed circular path of radius $\rho$ which encloses the straight conductor carrying direct current I. The point P is at a $\perp^r$ distance $\rho$ (r) from the conductor. Consider $d\overline{l}$ at a point P which is in $\overline{a_\phi}$ direction, tangential to circular path at point P

from biot-savart law

$$\overline{H} = \frac{I}{2\pi\rho}\overline{a_\phi}$$

$$\overline{H} \cdot d\overline{l} = \frac{I}{2\pi\rho}\overline{a_\phi} \cdot \rho d\phi \overline{a_\phi}$$

$$= \frac{I}{2\pi}d\phi$$

$$\oint \overline{H} \cdot d\overline{l} = \int_0^{2\pi} \frac{I}{2\pi}d\phi = \frac{I}{2\pi}[\phi]_0^{2\pi} = \frac{I\,\cancel{2\pi}}{\cancel{2\pi}}$$

$$\boxed{\Rightarrow \oint \overline{H} \cdot d\overline{l} = I}$$

## PERE'S CIRCUITAL LAW:-

onsider a closed path preferrably symmetrical such that it ncloses the direct current I once. This Amperian path onsider differential length $\bar{dl}$ depending upon the coordinate system used.

Identify the symmetry & find in which direction $\bar{H}$ exists according to the coordinate system used.

Find $\bar{H}\cdot\bar{dl}$, the dot product. Make sure that $\bar{dl}$ & $\bar{H}$ are in same direction

Find the integral of $\bar{H}\cdot\bar{dl}$ around the closed path assumed & equate it to current I enclosed by the path.

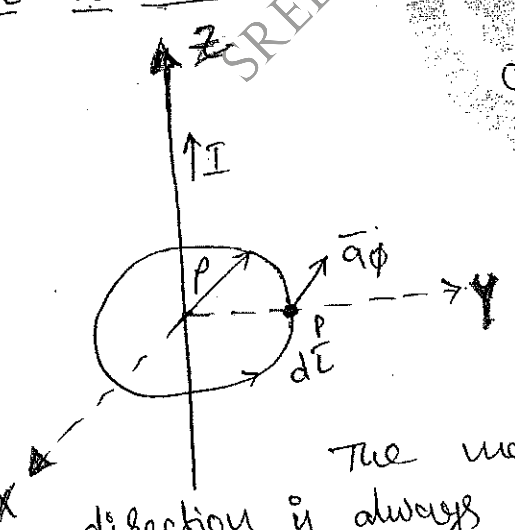
## APPLY AMPERE'S CIRCUITAL LAW:-

$\bar{H}$ is either tangential or normal to the path, at each point of the closed path.

The magnitude of $\bar{H}$ must be same at all points of the path where $\bar{H}$ is tangential.

## PLICATIONS OF AMPERE'S CIRCUITAL LAW:-

### DUE TO INFINITELY LONG STRAIGHT CONDUCTOR:-




Consider an infinitely long straight conductor placed along Z-axis, carrying a DC I as shown in fig.

Consider the amperian closed path, enclosing the conductor.

Consider point P is at a $\perp^r$ distance $\rho$ from the conductor.

The magnitude of $\bar{H}$ depends on $\rho$ and the direction is always tangential to the closed path i.e., $\bar{a}_\phi$ direction say $H_\phi$

so $\bar{H}$ has only component in $\bar{a}_\phi$ direction

Consider elementary length $d\bar{L}$ at point P and in cylinder coordinates it is $\rho d\phi$ in $\bar{a}_\phi$

$$\therefore \quad \bar{H} = H_\phi \bar{a}_\phi$$

$$d\bar{L} = \rho d\phi \bar{a}_\phi$$

$$\bar{H} \cdot d\bar{L} = H_\phi \bar{a}_\phi \cdot \rho d\phi \bar{a}_\phi$$

$$= H_\phi \rho d\phi$$

according to ampere's circuital law

$$\oint \bar{H} \cdot d\bar{L} = I$$

$$\int_0^{2\pi} H_\phi \rho d\phi = I$$

$$H_\phi \rho \int_0^{2\pi} d\phi = I$$

$$H_\phi \rho 2\pi = I$$

$$H_\phi = \frac{I}{2\pi\rho}$$

hence $\bar{H}$ at point P is given by

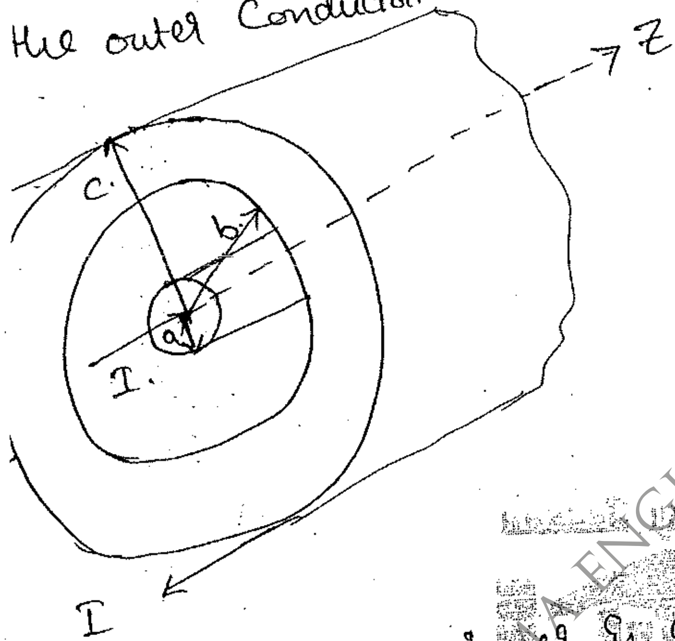
$$\bar{H} = H_\phi \bar{a}_\phi$$

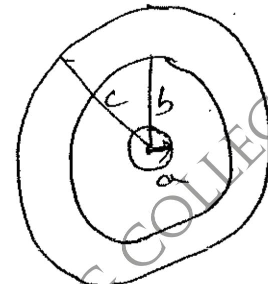
$$\boxed{\bar{H} = \frac{I}{2\pi r} \bar{a}_\phi \text{ A/m}}$$

7

## ...UE TO A COAXIAL CABLE:-

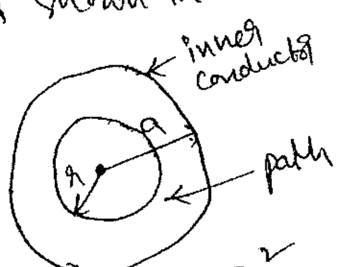
...onsider a co-axial cable, its inner conductor is solid with ...radius a, carrying I.
The outer conductor is in the form of concentric cylinder ...whose inner radius is b & outer radius is c.
This cable is placed along z-axis. The current I is uniformly ...distributed in the inner conductor. While $-I$ is uniformly distributed ...in the outer conductor.




The space b/w inner & outer conductor is filled with ...dielectric say air. The calculation of H is divided corresponding to ...various regions of the cable.

REGION - I:- within the inner conductor, $r<a$. Consider a closed path having radius $r<a$. hence it encloses only part of the conductor as shown in




The area of cross-section is $\pi r^2$ $m^2$.
The total current flowing is I through the area $\pi a^2$
hence the current enclosed by the closed path

$$I' = \frac{\pi r^2}{\pi a^2} I = \frac{r^2}{a^2} I$$

The $\bar{H}$ is again only in $\bar{a}_\phi$ direction & depends only on $r$

$$\bar{H} = H_\phi \bar{a}_\phi$$

Consider $d\bar{l}$ in the $\bar{a}_\phi$ direction which is $r\,d\phi$

$$d\bar{l} = r\,d\phi\,\bar{a}_\phi$$

$$\bar{H}\cdot d\bar{l} = H_\phi \bar{a}_\phi \cdot r\,d\phi\,\bar{a}_\phi$$

$$= H_\phi\, r\,d\phi$$

according to ampere's circuital law,

$$\oint \bar{H}\cdot d\bar{l} = I'$$

$$\oint H_\phi\, r\,d\phi = \frac{r^2}{a^2} I$$

$$\int_0^{2\pi} H_\phi\, r\,d\phi = \frac{r^2}{a^2} I$$

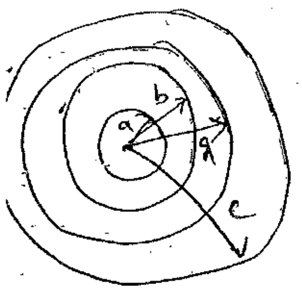
$$H_\phi = \frac{r^2}{2\pi r a^2} I$$

$$\boxed{\bar{H} = \frac{I r}{2\pi a^2}\bar{a}_\phi \text{ A/m}}$$ --- $r < a$ within conductor

REGION 2: within $a < r < b$ Consider a circular path which encl the inner conductor carrying DC I. This is the case of infinitely lon conductor along z-axis hence $\bar{H}$ in this region

$$\boxed{\bar{H} = \frac{I}{2\pi r}\bar{a}_\phi \text{ A/m}}$$ --- $a < r < b$

REGION-3: $b < r < c$



Consider the closed path, the current enclosed by the closed path is only the part of the current $-I$, in the outer conductor.

The total current $-I$ is flowing through the cross-section $\pi(c^2-b^2)$ while the closed path enclose the cross section $\pi(r^2-b^2)$

hence the current enclosed by the closed path of outer conductor is

$$I' = \frac{\pi(r^2-b^2)}{\pi(c^2-b^2)}(-I)$$

$$= -\frac{(r^2-b^2)}{(c^2-b^2)} I$$

$I'' = I =$ current in inner conductor enclosed

$$I_{enc} = I' + I'' = -\frac{r^2-b^2}{c^2-b^2} I + I$$

$$\frac{-r^2+\cancel{b^2}+c^2-\cancel{b^2}}{c^2-b^2}$$

$$= I\left[\frac{c^2-r^2}{c^2-b^2}\right]$$

$$\oint \bar{H}\cdot d\bar{l} = I_{enc}$$

$\bar{H} = H_\phi \bar{a}_\phi$, $d\bar{l} = r\,d\phi\, \bar{a}_\phi$

$$\int_0^{2\pi} H_\phi\, r\,d\phi = I_{enc}$$

$$\boxed{\bar{H} = H_\phi \bar{a}_\phi = \frac{I}{2\pi r}\left(\frac{c^2-r^2}{c^2-b^2}\right)\bar{a}_\phi \ \text{A/m}}$$ --- $b<r<c$

REGION 4: $r > c$

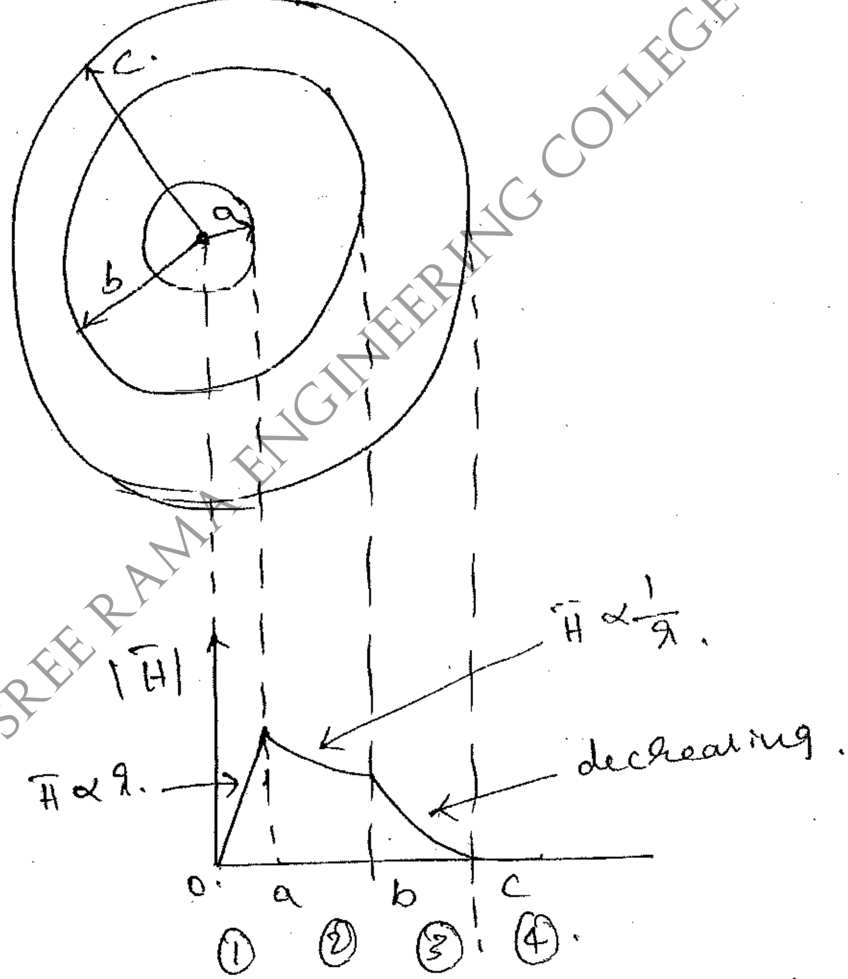
Consider the closed path with $r > c$ such that it encloses both the conductors i.e., both $+I$ & $-I$

$$I_{enc} = +I - I = 0$$

$$\oint \bar{H} \cdot d\bar{l} = 0$$

$$\boxed{\bar{H} = 0 \text{ A/m}} \quad r > c$$

The magnetic field doesnot exist outside the cable.




(*) obtain the expression for $\bar{H}$ in all the regions if a cylindrical conductor carries a direct current $I$ and its radius is $R$ m. plot the variation of $\bar{H}$ against the distance $r$ from the centre.

$\bar{H} = H_\phi \bar{a}_\phi$

$\bar{dl} = r d\phi \bar{a}_\phi$

$\oint \bar{H} \cdot d\bar{l} = I \frac{r^2}{R^2}$

$H_\phi = \frac{Ir}{2\pi R^2}$

$\bar{H} = \frac{Ir}{2\pi R^2} \bar{a}_\phi \; (r < R)$

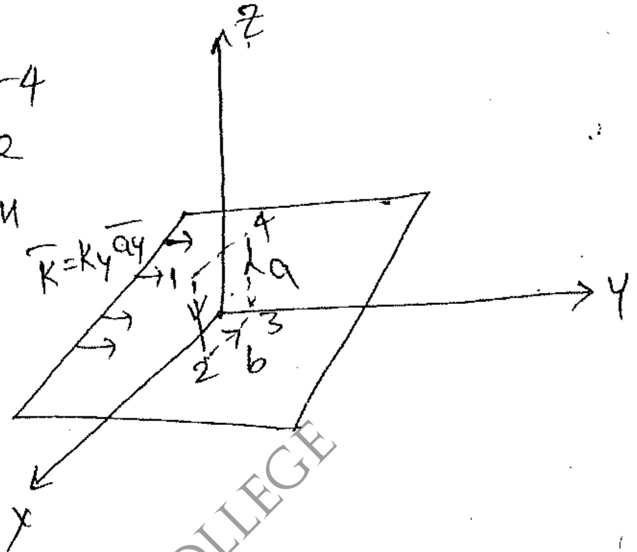
on the surface $r = R$

$\bar{H} = \frac{I}{2\pi R} a_\phi$

$\bar{H} = \frac{I}{2\pi r}$

## $\ddagger$ DUE TO INFINITE SHEET OF CURRENT:- 

Consider an infinite sheet of current in the $z=0$ plane. The surface current density is $\bar{K}$. The current is flowing in positive y direction hence $\bar{K} = K_y \bar{a}_y$.

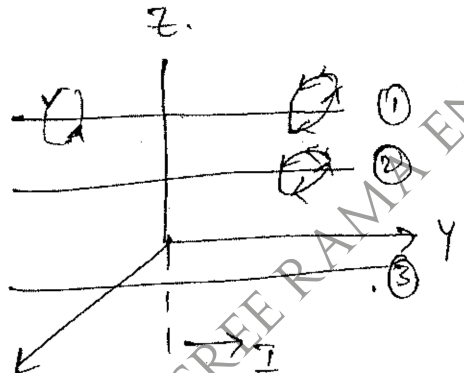


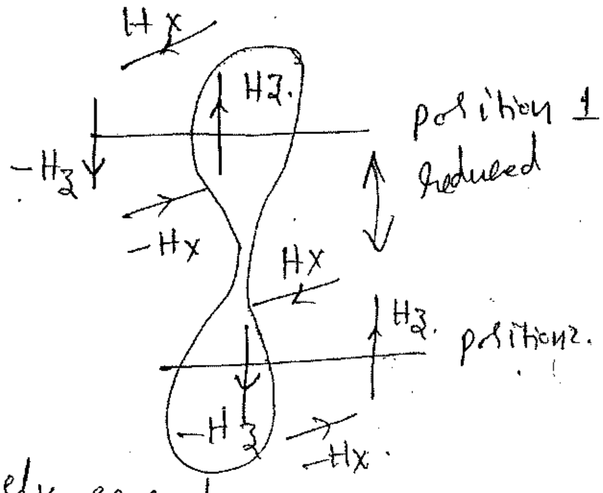

Consider a closed path 1-2-3-4 the width of the path is 'b' while the height is 'a'. It is $\perp^r$ to the direction of current hence in XZ plane

The current flowing across a distance b is given by $K_y b$

$\therefore I_{enc} = K_y b$

Consider the magnetic lines of force due to the current in $\bar{a}_y$ direction, according to right hand thumb rule.

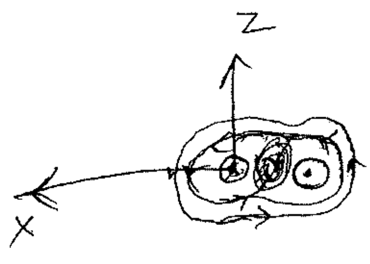



It is clear that b/w 2 very closely spaced conductors, the components of $\bar{H}$ in z-direction are oppositely directed $\therefore$ $\bar{H}$ cannot have any component in z direction.

as current is flowing in y direction $\bar{H}$ cannot have any component in y direction

$$\therefore \bar{H} = H_x \bar{a}_x \quad \text{for } z>0$$
$$= -H_x \bar{a}_x \quad \text{for } z<0$$



applying ampere's circuital law

$$\oint \bar{H}\cdot d\bar{L} = I_{enc}$$

Integral along the path 1-2-3-4-1

for path 1-2, $d\bar{L} = dz\,\bar{a}_z$

3-4 $d\bar{L} = dz\,\bar{a}_z$

~~in x-direction 1-2, 3-4 $\Rightarrow \oint \bar{H}\cdot d\bar{L} = 0$, $\bar{a}_x\cdot\bar{a}_z = 0$~~

but $\bar{H}$ is in x-direction $\oint \bar{H}\cdot d\bar{L} \Rightarrow \bar{a}_x\cdot\bar{a}_z = 0$.

$\therefore$ for path 1-2 & 3-4, $\oint \bar{H}\cdot d\bar{L} = 0$

path 2-3 $d\bar{L} = dx\,\bar{a}_x$

$$\int_2^3 \bar{H}\cdot d\bar{L} = \int_2^3 (-H_x\bar{a}_x)\cdot dx(\bar{a}_x) = H_x\int_2^3 dx = bH_x$$

path 2-3 is lying in $z<0$ region for which $\bar{H}$ is $-H_x\bar{a}_x$ & limit from 2 to 3, +ve x to -ve x hence effective sign of the integral is +ve.

path 4-1 along which $d\bar{L} = dx\,\bar{a}_x$ & it is in the region $z>0$

hence $\bar{H} = H_x\bar{a}_x$

$$\therefore \int_4^1 \bar{H}\cdot d\bar{L} = \int_4^1 (H_x\bar{a}_x)\cdot(dx\,\bar{a}_x) = H_x\int_4^1 dx = bH_x$$

$$\therefore \oint \bar{H}\cdot d\bar{L} = bH_x + bH_x = 2bH_x.$$

hence $\bar{H} = \frac{1}{2}K_y\bar{a}_x \quad z>0$

$= -\frac{1}{2}K_y\bar{a}_x \quad z<0.$

$$2bH_x = K_y b$$

$$H_x = \frac{1}{2}K_y$$

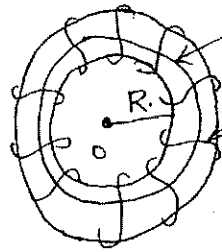
In general, for an infinite sheet of current density $\bar{K}$ A/m

$$\boxed{\bar{H} = \frac{1}{2}\bar{K}\times\bar{a}_N}$$

## APPLICATIONS OF AMPERE'S LAW:

Consider a troidal coil of N turns, when a current flows in the coil, it sets up magnetic flux lines concentric to the axis of the troidal ring, but confined within the hollow space inside the turns, ie., the field is zero outside the troid.

→ Now choose a circular closed path which runs within the body of the ring, the path encloses all the N turns of the coil through each of which a current I flows.




APPLICATIONS:

1. high frequency applications.
2. boosts the freq to appropriate values.
3. Electronic applications

applying ampere's circuital law

$$\oint \overline{H}\cdot\overline{dl} = NI$$

$$\oint H\,dl\cos\theta = NI$$

→ Let the thickness of the troidal coil be small compared to its radius R. Therefore H remains const. over the coil.

$$\therefore H\oint dl\cos\theta = NI$$

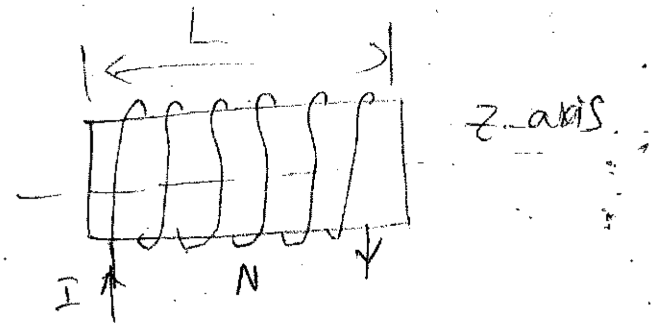
∵ the closed path is a circle of radius R

$$\oint dl\cos\theta = 2\pi R$$

$$H\cdot 2\pi R = NI$$

$$H = \frac{NI}{2\pi R} \quad \text{or} \quad B = \frac{\mu NI}{2\pi R}$$

## $\bar{H}$ for a Solenoid:




$\bar{H} = H_z \bar{a_z}$

$\bar{dL} = dz\, \bar{a_z}$

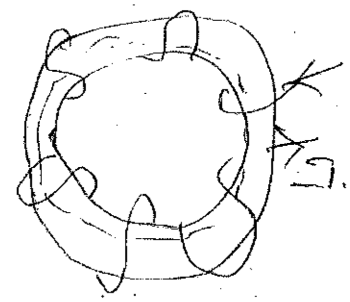
$\oint \bar{H} \cdot \bar{dL} = I_{encl}$

$\oint H_z\, dz\, \bar{a_z} \cdot \bar{a_z} = NI$

$H_z [dz]_0^L = NI$

$H_z = \frac{NI}{L} \bar{a_z}$

## $\bar{H}$ for a Toroid:




$\bar{H} = H_\phi \bar{a_\phi}$

$\bar{dL} = \rho\, d\phi\, \bar{a_\phi}$

$\oint \bar{H} \cdot \bar{dL} = I_{encl}$

$\oint_0^{2\pi} \bar{H}_\phi \rho\, d\phi = NI$

$H_\phi = \frac{NI}{2\pi\rho}$

$\bar{H} = \frac{NI}{2\pi\rho} \bar{a_\phi}$

$$\therefore dH = \frac{NI R^2 (-R \operatorname{Cosec}^2\phi) d\phi}{2l \; R^3 \operatorname{cosec}^3\phi} = -\frac{NI}{2l} \sin\phi \; d\phi$$

$\therefore$ H at P is $H = \int dH$

$$= \int_{\phi_1}^{\phi_2} -\frac{NI}{2l} \sin\phi \; d\phi$$

$$= -\frac{NI}{2l} \Big[-\cos\phi\Big]_{\phi_1}^{\phi_2}$$

$$= \frac{NI}{2l} \left[\cos\phi_2 - \cos\phi_1\right]$$

Case ①:- $\vec{H}$ due to long solenoid :

$\phi_1 = \pi$ and $\phi_2 = 0$

$$\therefore \vec{H} = \frac{NI}{2l}\left[\cos 0 - \cos\pi\right] = \frac{NI}{l} \vec{a_x}$$

Case ②:- $\vec{H}$ ~~due~~ at the end of a long solenoid :

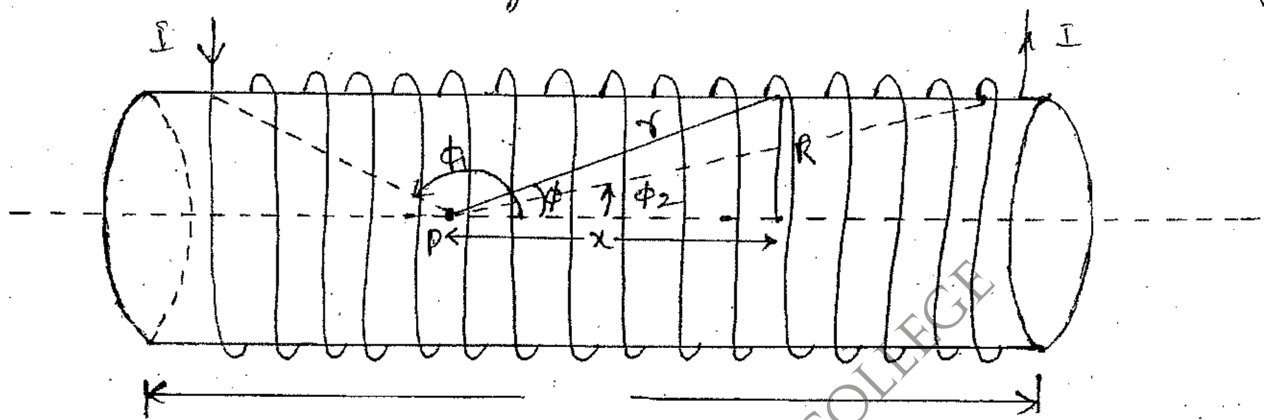
then $\phi_1 = \frac{\pi}{2}$ and $\phi_2 = 0$

$$\therefore \vec{H} = \frac{NI}{2l}\left[\cos 0 - \cos\frac{\pi}{2}\right] = \frac{NI}{2l} \vec{a_x}$$

Where $\vec{a_x}$ is a unit vector pointing from ~~left~~ left to Right along the axis, for the direction of current shown in fig.

## Magnetic field Intensity at a point on the Axis of a Solenoid :-

Consider a solenoid of radius R with N turns and of length '$l$', carrying a current I as shown in fig.




Consider a point P on its axis. Also consider a differential element of thickness dx of the solenoid which will be in the form of a coil of $n$ turns whose centre is at a distance $x$ from P.

The no. of turns in the coil (element) $= n = \left(\frac{N}{l}\right) dx$

$\therefore$ The magnetic field Intensity due to this element (coil) at P is

$\sqrt{R^2+x^2} = R\csc\phi$

$$dH = \frac{\left(\frac{N}{l}\right) dx\, I R^2}{2\left(R^2+x^2\right)^{3/2}}$$

$\sqrt{R^2+x^2}$ $(R^2+x^2)^{1/2} = (\sqrt{r})^2 = r$.

Let $x = R\cot\phi$

Cubing $R^2+x^2 = r^2$

$dx = -R\,\operatorname{cosec}^2\phi\, d\phi$

$(R^2+x^2)^{3/2} = r^3$

Let $R^2+x^2 = r^2$ $\quad \left(\sqrt{R^2+x^2}\right)^3 = r^3$

$(R\operatorname{cosec}\phi)^3 = r^3$

$\sqrt{R^2+x^2} = r$ $\quad (R\csc\phi)^3 = r^3$

Cubing

$$dH = \frac{NI\rho^2(-\rho\csc^2\phi)d\phi}{2l(\rho\csc\phi)^3}$$

$$= -\frac{NI}{2l}\sin\phi \, d\phi$$

$\therefore$ H at P is, $H = \int dH$

$$= -\frac{NI}{2l}\int_{\phi_1}^{\phi_2} \sin\phi \, d\phi$$

$$= -\frac{NI}{2l}\left[-\cos\phi\right]_{\phi_1}^{\phi_2}$$

$$= \frac{NI}{2l}\left[\cos\phi_2 - \cos\phi_1\right]$$

APPLICATIONS:

1. Locking device, door locks for offices, hotels, high security areas.
2. Dialysis machine, 2 solenoids to control a person's blood flow
3. Interlocking in the gearbox drives
4. locking, positioning holding, rotating valve operational
5. Control water pressure in the sprinklers system.

(1) H due to a long solenoid

If the solenoid is of infinite length, then $\phi_1 = \pi$, $\phi_2 = 0$

$$\bar{H} = \frac{NI}{2l}\left[\cos 0 - \cos\pi\right]$$

$$\boxed{\bar{H} = \frac{NI}{l}\bar{a}_z} \rightarrow (1)$$

(2) H at the end of a long solenoid,

$\phi_1 = \pi/2$ & $\phi_2 = 0$

$$\therefore H = \frac{NI}{2l}\left(\cos 0 - \cos\pi/2\right)$$

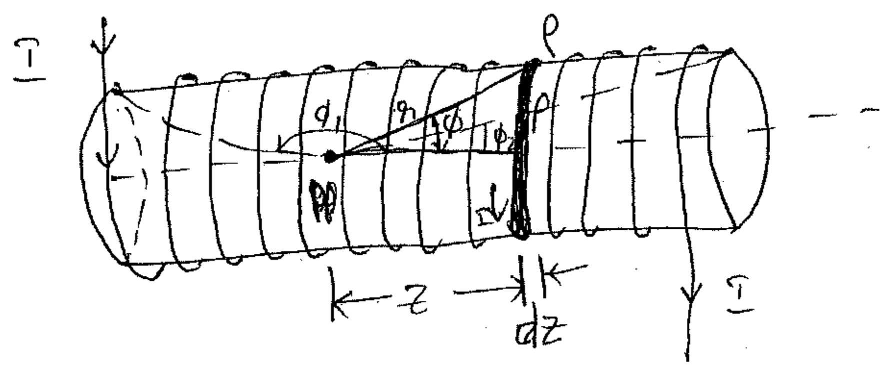
$$\boxed{H = \frac{NI}{2l}} \rightarrow (2)$$

Comparing (1) & (2).

$\bar{H}$ at the end of a long solenoid is one half of that at the center.

## H at a point on the Axis of a SOLENOID:-






$\rho = R$

$x = z$

$$\bar{H} = \frac{I \rho^2}{2(\rho^2 + z^2)^{3/2}} \Rightarrow \frac{I R^2}{2(R^2 + x^2)^{3/2}} \quad \left(\bar{H} \text{ on the axis of a circular loop}\right)$$

Consider a solenoid of radius $\rho$ with N turns & of length $l$, carrying a current $I$. Consider a point P on its axis. Also, consider a differential element of thickness $dz$ of the solenoid which will be in the form of a coil of $n$ turns, whose center is at a distance $z$ from P.

The no. of turns in the coil, $n = \left(\frac{N}{l}\right) dz$.

$\therefore$ $\bar{H}$ due to this coil at P is

$$dH = \frac{\left(\frac{N}{l}\right) dz \, I \rho^2}{2(\rho^2 + z^2)^{3/2}}$$

$z = \rho \cot\phi$

$\rho^2 + z^2 = r^2$

$dz = -\rho \csc^2\phi \, d\phi$

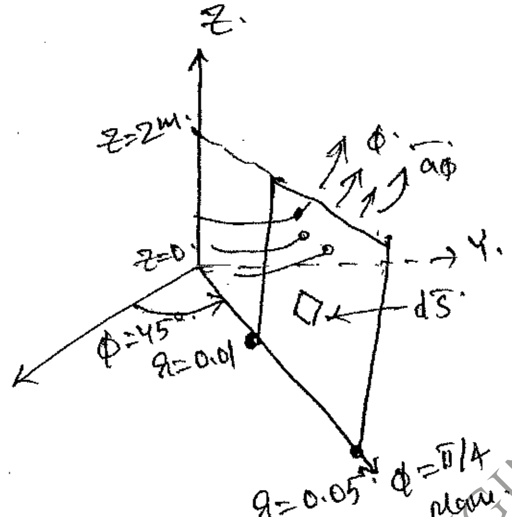
$(\rho^2 + z^2)^{3/2} = r^3 = (\rho \csc\phi)^3$

$$= \int_{z=0}^{d} \int_{r=a}^{b} \frac{\mu_0 I}{2\pi r} \, dr \, dz$$

$$\phi = \frac{\mu_0 I d}{2\pi} \ln\left[\frac{b}{a}\right] \text{ wb}$$

Q) Find the flux passing the portion of the plane $\phi = \pi/4$ defined by $0.01 < r < 0.05m$ and $0 < z < 2m$. A current filament of 2.5A is along the z-axis in the $\bar{a}_z$ direction, in free space.

A:




due to current carrying conductor in free space along z-axis,

$$\bar{H} = \frac{I}{2\pi r} \bar{a}_\phi$$

$$= \frac{2.5}{2\pi r} \bar{a}_\phi$$

$$= \frac{0.3978}{r} \bar{a}_\phi$$

$$\bar{B} = \mu_0 \bar{H} = \frac{4\pi \times 10^{-7} \times 0.3978}{r} \bar{a}_\phi$$

$$= \frac{5 \times 10^{-7}}{r} \bar{a}_\phi \text{ wb/m}^2$$

the flux crossing the surface is

$$\phi = \int_s \bar{B} \cdot d\bar{S}$$

$d\bar{S} = dr\,dz\,\bar{a}_\phi$ normal to $\bar{a}_\phi$

$$\phi = \int_{z=0}^{2} \int_{r=0.01}^{0.05} \frac{5 \times 10^{-7}}{r} \bar{a}_\phi \cdot dr\,dz\,\bar{a}_\phi$$

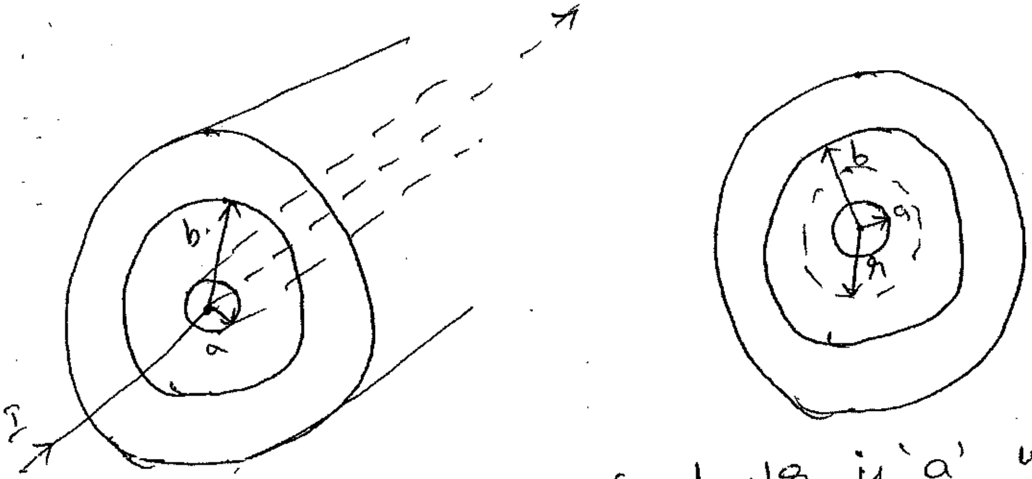
$$= 5 \times 10^{-7} \left[\ln r\right]_{0.01}^{0.05} \left[z\right]_0^2$$

$$= 5 \times 10^{-7} \times 2 \times \ln\left[\frac{0.05}{0.01}\right]$$

$$= 1.6094 \ \mu\text{wb}$$

## APPLICATION OF FLUX DENSITY & FLUX TO CO-AXIAL CABLE:-






The radius of the inner conductor is 'a' while the inner radius of the outer conductor is 'b'.

It carries a DC I which is uniformly distributed in the inner conductor. The outer conductor carries same current I in opposite direction to that carried by the inner conductor.

$\bar{H}$ in the region $a < r < b$

$$\bar{H} = \frac{I}{2\pi r} \bar{a}_\phi \ A/m$$

The cable is filled with the air as dielectric with $\mu = \mu_0$

$$\bar{B} = \mu_0 \bar{H} = \frac{\mu_0 I}{2\pi r} \bar{a}_\phi \ wb/m^2$$

Let 'd' be the length of the conductors. The magnetic flux contained b/w the conductors in a length 'd' is the magnetic flux crossing the radial plane from $r = a$ to $r = b$ & for $z = 0$ to $z = d$.

$$\phi = \int_S \bar{B} \cdot d\bar{S}$$

$d\bar{S}$ normal to the $\bar{a}_\phi$ direction is $dr\,dz$

$$d\bar{S} = dr\,dz\,\bar{a}_\phi$$

$$\phi = \int_S \bar{B} \cdot d\bar{S} = \int_S \frac{\mu_0 I}{2\pi r} \bar{a}_\phi \cdot dr\,dz\,\bar{a}_\phi$$

UNIT-IV



## ORCES DUE TO MAGNETIC FIELDS:-

There are three ways in which force due to Magnetic fields can be experienced. The force can be

* due to a moving charged particle in a magnetic field
* On a Current Element in an External magnetic field.
* between two Current Elements.

## 1. FORCE ON A CHARGED PARTICLE:-

* The Electric field Causes a force to be exerted on a Charge which may be either Stationary or in motion
* the Steady magnetic field is capable of exerting a force only on a moving Charge.
* A Magnetic field may be produced by moving charges and may exert forces on moving charges.
* A magnetic field cannot arise from Stationary Charges and cannot exert any force on a Stationary Charge.

The Electric force $\vec{F_e}$ on a Stationary or moving Electric charge Q in a Electric field is given by Coulomb's experimental Law and is related to the Electric field intensity $\vec{E}$ as

$$\vec{F_e} = Q\vec{E} \longrightarrow (1)$$

The eq → ① shows that if Q is positive then $\vec{F_e}$ and $\vec{E}$ have the same direction.

* A magnetic field can exert force only on a moving charge. The magnetic force $\vec{F_m}$ experienced by a charge Q moving with a velocity $\vec{u}$ in a magnetic field $\vec{B}$ is

$$\vec{F_m} = Q\vec{u} \times \vec{B} \longrightarrow ②$$

This clearly shows that $\vec{F_m}$ is perpendicular to both $\vec{u}$ and $\vec{B}$.

* Compare eq → ① and eq → ②, the following points are noted.

→ $\vec{F_e}$ is independent of the velocity of the charge and can perform work on the charge and change its kinetic energy.

→ $\vec{F_m}$ depends on the charge velocity and is normal to it. $\vec{F_m}$ cannot perform work because it is at right angles to the direction of motion of the charges. It does not cause an increase in kinetic energy of the charge.

for a moving charge Q in the presence of both Electric and magnetic fields, the Total force on the charge is given by

$$\vec{F} = \vec{F_e} + \vec{F_m}$$

$$\vec{F} = Q\left(\vec{E} + \vec{u} \times \vec{B}\right) \longrightarrow (3)$$

This is known as Lorentz force equation.

i) FORCE ON A CURRENT ELEMENT:-

* A current Element is imagined in Magnetostatics in analogy to the Electric Charge in Electrostatics.
* An Electric field arises due to an Electric Charge.
* A magnetic field is imagined to be produced due to a differential element of current i.e due to the current flowing in a differential element of the Conductor.
* If the differential Element is of length $dL$ through which a current $I$ is flowing, then the product $IdL$ represents the Current Element

* A current flow means flow of charges. When a current carrying conductor is placed in magnetic field, the moving charges experience the Lorentz force. Since the charges are confined inside the conductor, the force experienced by the charges act on the conductor. As a result, the conductor itself experiences a force.

* The relationship between line current, surface current and volume current elements are

$$I\,\vec{dl} = \vec{K}\,ds = \vec{J}\,dv$$

Here $I\,\vec{dl} = \frac{dQ}{dt}\vec{dl} = dQ\,\frac{\vec{dl}}{dt} = dQ\,\vec{u}$

$$\therefore\ I\,\vec{dl} = dQ\,\vec{u} \longrightarrow$$

This shows that an elemental charge $dQ$ moving with velocity $\vec{u}$ is equivalent to $I\,\vec{dl}$. Thus the force on a current element $I\,\vec{dl}$ in a magnetic field $\vec{B}$ is found from eq→② by merely replacing $Q\vec{u}$ by $I\,\vec{dl}$ i.e

$$\vec{dF} = I\,\vec{dl} \times \vec{B}$$

* If the current I is through a closed path L or circuit, the force on the circuit is given by

$$\vec{F} = \oint_L I\, d\vec{L} \times \vec{B} \quad (\text{for line current})$$

$$\vec{F} = \oint_S \vec{K}\, ds \times \vec{B} \quad (\text{for surface current})$$

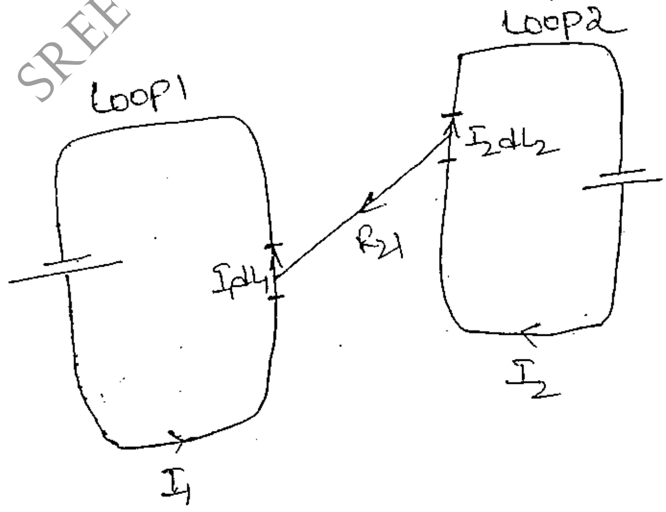
$$\vec{F} = \int_V \vec{J}\, dv \times \vec{B} \quad (\text{for volume current})$$

* The magnetic field $\vec{B}$ is defined as the force per unit current element.



) FORCE BETWEEN TWO CURRENT ELEMENTS :-

Consider the force between two current elements $I_1 dL_1$ and $I_2 dL_2$ as shown in fig.




~~According~~ to Biot-Savart's law, both current elements produce magnetic fields

The force $d(dF_1)$ on element $I_1 dL_1$ due to field $dB_2$ produced by element $I_2 dL_2$ as shown in fig.

$$d(\vec{dF_1}) = I_1 \vec{dL_1} \times \vec{dB_2}$$

But from Biot-Savart's Law,

$$\vec{dB_2} = \frac{\mu_0 I_2 \vec{dL_2} \times \vec{a}_{R_{21}}}{4\pi R_{21}^2}$$

$$\therefore \quad d(\vec{dF_1}) = \frac{\mu_0 I_1 \vec{dL_1} \times (I_2 \vec{dL_2} \times \vec{a}_{R_{21}})}{4\pi R_{21}^2}$$

This equation is essentially the law of force between two current elements and is analogous to Coulomb's Law, which expresses the force b/n two stationary charges.

$\therefore$ The Total force $\vec{F_1}$ on current loop 1 due to current loop 2 is

$$\vec{F_1} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{\vec{dL_1} \times (\vec{dL_2} \times \vec{a}_{R_{21}})}{R_{21}^2}$$

# FORCE ON A STRAIGHT CURRENT CARRYING CONDUCTOR IN A MAGNETIC FIELD :-

The force on a differential current element in a magnetic field as

$$\vec{dF} = I(\vec{dl} \times \vec{B})$$

Instead of a current element, if a current carrying straight conductor of length $l$ in a uniform magnetic field of flux density $\vec{B}$ is considered, then the force $\vec{F}$ acting on it can be written as

$$\vec{F} = \int I(\vec{dl} \times \vec{B}) = I(\vec{L} \times \vec{B})$$

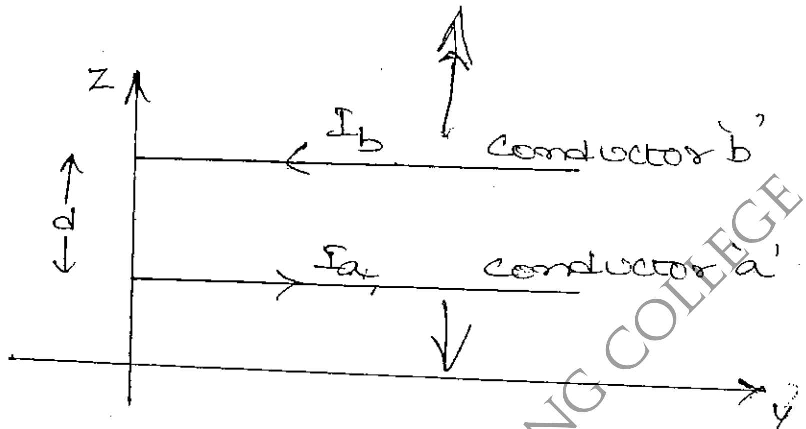
If the magnetic field's direction is $\perp$r to the direction of flow of current then $\theta = \pi/2$ and the magnitude of the the force on the conductor becomes

$$F = ILB \sin\theta$$

$$F = ILB \quad (\because \theta = \pi/2, \ \sin \tfrac{\pi}{2} = 1)$$

## FORCE BETWEEN TWO STRAIGHT LONG PARALLEL CONDUCTORS CARRYING CURRENT IN OPPOSITE DIRECTIONS:-

Let the currents through conductors be flowing in reverse direction, with $I_a$ towards positive Y-axis and $I_b$ towards -ve Y-axis as shown in fig.




The magnetic flux density at conductor 'b' due to current $I_a$ through conductor (a) is

$$\vec{B_a} = \frac{\mu_0 I_a}{2\pi d}\vec{a_x}$$

∴ force applied on conductor (b) by the field of conductor (a) is

$$\vec{f_a} = I_b \vec{L} \times \vec{B}$$

$$= I_b L(-\vec{a_y}) \times \frac{\mu_0 I_a}{2\pi d}\vec{a_x}$$

$$= \frac{\mu_0 I_a I_b L}{2\pi d}\vec{a_z}$$

The magnetic flux density at conductor (b) due to current ($I_b$) through conductor (a)

$$\vec{B_b} = \frac{\mu_0 I_b}{2\pi d}\vec{a_x}$$

force applied on conductor (B) by the field of conductor (b),

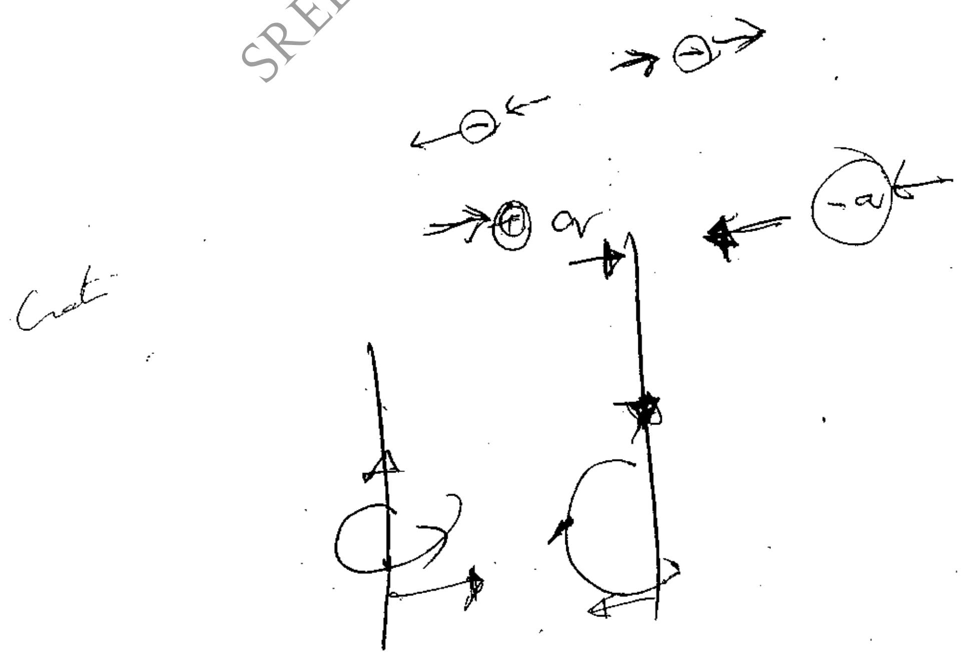
$$\vec{F_b} = I_a \vec{L} \times \vec{B_b}$$

$$= I_a L \vec{a_y} \times \frac{\mu_0 I_b}{2\pi d} \vec{a_x}$$

$$= -\frac{\mu_0 I_a I_b L}{2\pi d} \vec{a_z}$$

As both the forces $\vec{F_a}$ and $\vec{F_b}$ are acting in opposite directions, the force between the conductors is a Repulsive force.

* When the currents direction are same then the forces $\vec{F_a}$ and $\vec{F_b}$ are acting in the same direction. It is an Attractive force directed normal to the plane in which the conductors are present.



AGNETIC TORQUE:-

$$\bar{F} = \oint I\, d\bar{L} \times \bar{B}$$

$$= -I \oint \bar{B} \times d\bar{L}$$

If magnetic flux density is uniform,

$$\bar{F} = -I\bar{B} \times \oint d\bar{L}$$

but for a closed ckt, $d\bar{L} = 0$ thus the force on a closed lamentary ckt is zero in the uniform magnetic field. the field is not uniform, then force on closed ckt is not zero.

MAGNETIC TORQUE:- It is defined as the vector product of the moment arm $\bar{R}$ and the force $\bar{F}$. It is measured in newton meter.

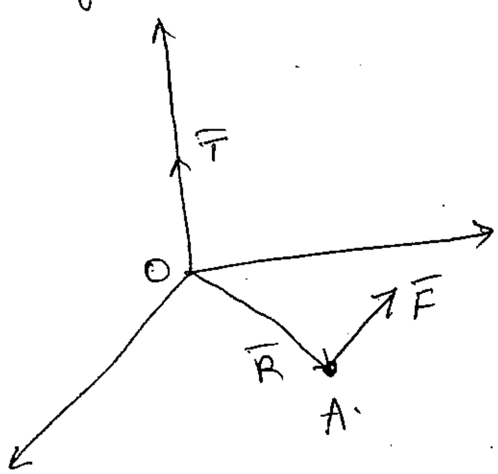
$$\bar{T} = \bar{R} \times \bar{F} \ \text{N-m.}$$

Consider a point A at which force $\bar{F}$ is applied, let $\bar{R}$ be the arm from origin O at point A.

then the torque $\bar{T}$ about origin is $\bar{R} \times \bar{F}$.

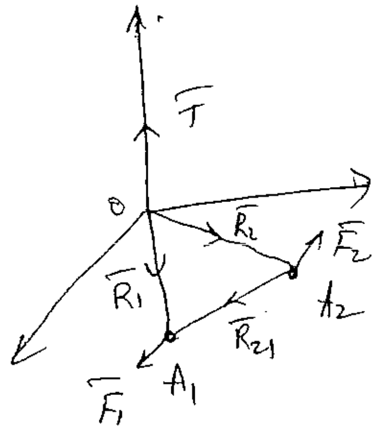
The magnitude of the torque is equal to the product of magnitudes of $\bar{R}$ & $\bar{F}$ & sine of the angle b/w $\bar{R}$ & $\bar{F}$,

the direction of the $\bar{T}$ is normal to both $\bar{R}$ and $\bar{F}$



Now Consider that two forces namely $\bar{F_1}$ & $\bar{F_2}$ are applied at points $A_1$ & $A_2$, the arms for the two forces drawn from the origin be $\bar{R_1}$ & $\bar{R_2}$

assume that $\bar{F_2} = -\bar{F_1}$




$$\bar{T} = \bar{R_1} \times \bar{F_1} + \bar{R_2} \times \bar{F_2}$$

$$\bar{T} = (\bar{R_1} - \bar{R_2}) \times \bar{F_1} \quad \therefore \bar{F_2} = -\bar{F_1}$$

$$\bar{T} = \bar{R_{21}} \times \bar{F_1}$$

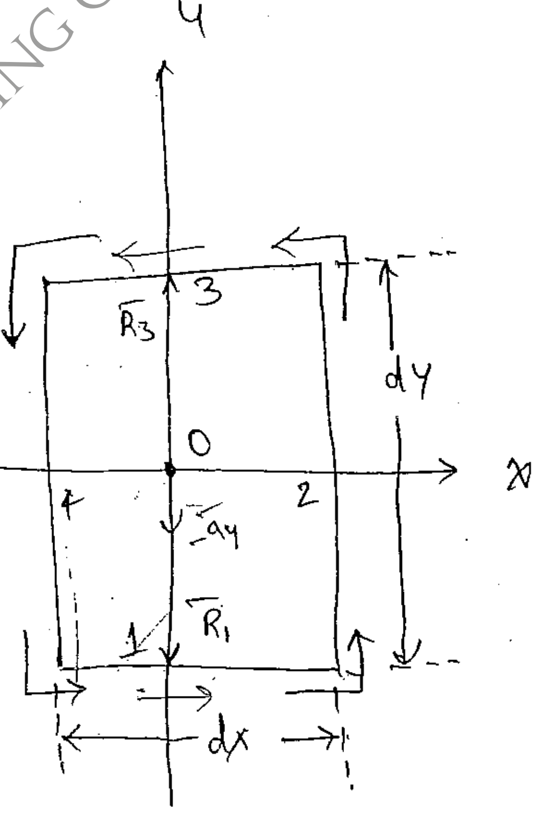
$$\bar{R_{21}} = \bar{R_1} - \bar{R_2}$$

## TORQUE ON A CURRENT LOOP PLACED IN MAGNETIC FIELD:-

Consider a differential current loop placed in x-y plane in the magnetic field $\bar{B}$. The loop is placed in the plane such that the sides of the loop are parallel to the axes resp.

Let dx & dy be the lengths of the sides of the loop as shown in fig.

assume that the current in loop flows in anticlockwise direction.




The differential force exerted on side 1

$$d\bar{F_1} = I\,dL_1 \times \bar{B_0}$$

$$= I\,dx\,\bar{a_x} \times \bar{B_0}$$

$$\bar{B_0} = B_{0x}\bar{a_x} + B_{0y}\bar{a_y} + B_{0z}\bar{a_z}$$

$$\begin{vmatrix} \bar{a_x} & \bar{a_y} & \bar{a_z} \\ I\,dx & 0 & 0 \\ B_{0x} & B_{0y} & B_{0z} \end{vmatrix}$$

$$d\bar{F_1} = I\,dx\,(B_{0y}\bar{a_z} - B_{0z}\bar{a_y}).$$

$$\bar{a_x}(0) - \bar{a_y}(I\,dx\,B_{0z}) + \bar{a_z}(I\,dx\,B_{0y})$$

$$\therefore d\bar{F_1} = I\,dx\,(B_{0y}\bar{a_z} - B_{0z}\bar{a_y})$$

The moment arm for this side is the arm or vector which ends from origin to the midpoint of the side.

$$\bar{R}_1 = -\frac{1}{2} dy\, \bar{a}_y$$

$\therefore$ torque on side 1

$$d\bar{T}_1 = \bar{R}_1 \times d\bar{F}_1$$
$$= -\frac{1}{2} dy\, \bar{a}_y \times I dx (B_{oy} \bar{a}_z - B_{oz} \bar{a}_y)$$
$$= -\frac{1}{2} I dx dy (B_{oy} \bar{a}_x)$$

for side 3

$$d\bar{F}_3 = I d\bar{L}_3 \times \bar{B}_0$$
$$= -I (dx\, \bar{a}_x) \times (B_{ox} \bar{a}_x + B_{oy} \bar{a}_y + B_{oz} \bar{a}_z)$$
$$= -I dx (B_{oy} \bar{a}_z - B_{oz} \bar{a}_y)$$

$$\Rightarrow \bar{R}_3 = \frac{1}{2} dy\, \bar{a}_y$$

$$d\bar{T}_3 = \bar{R}_3 \times d\bar{F}_3$$
$$= \frac{1}{2} dy\, \bar{a}_y \times (-I dx (B_{oy} \bar{a}_z - B_{oz} \bar{a}_y))$$
$$= -\frac{1}{2} I dx dy\, B_{oy}\, \bar{a}_x$$

The torque contribution of sides 1 & 3 are same.

$$\therefore d\bar{T}_1 + d\bar{T}_3 = -I dx dy\, B_{oy}\, \bar{a}_x$$

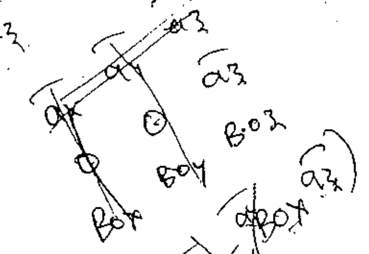
Illy for side 2

$$d\bar{T}_2 = \bar{R}_2 \times d\bar{F}_2$$
$$= \left(\frac{1}{2} dx\, \bar{a}_x\right) \times \left( (I dy\, \bar{a}_y) \times (B_{ox} \bar{a}_x + B_{oy} \bar{a}_y + B_{oz} \bar{a}_z) \right)$$
$$= -\frac{1}{2} dx dy\, I (-B_{ox} \bar{a}_y)$$
$$= \frac{1}{2} dx dy\, I B_{ox}\, \bar{a}_y$$

$$d\bar{T}_4 = \frac{1}{2} dx\,dy\, I B_{0x}\, \bar{a}_y$$

$$d\bar{T}_2 + d\bar{T}_4 = dx\,dy\, I B_{0x}\, \bar{a}_y$$

$\bar{B}_0 \times \bar{a}_z$



$$d\bar{T} = d\bar{T}_1 + d\bar{T}_2 + d\bar{T}_3 + d\bar{T}_4$$

$$= I\,dx\,dy\left(B_{0x}\bar{a}_y - B_{0y}\bar{a}_x\right)$$

but $\left(B_{0x}\bar{a}_y - B_{0y}\bar{a}_x\right) = \bar{a}_z \times \left(B_{0x}\bar{a}_x + B_{0y}\bar{a}_y + B_{0z}\bar{a}_z\right)$

$$d\bar{T} = I\,dx\,dy\left(\bar{a}_z \times \bar{B}_0\right)$$

but $(dx\,dy)\,\bar{a}_z$ as $d\bar{S}$ which is called as vector area of the differential current loop

$$\boxed{\therefore\ d\bar{T} = I\, d\bar{S} \times \bar{B}}$$

MAGNETIC DIPOLE MOMENT:-

The magnetic dipole moment of a current loop is defined as the product of current through the loop and the area of the loop, directed normal to the current loop. denoted by '$\bar{m}$' the direction of $\bar{m}$ is given by the right hand thumb rule.

$$\bar{m} = (IS)\,\bar{a}_n \quad A\cdot m^2$$

using definition for the magnetic dipole moment,

$$\bar{T} = \bar{m} \times \bar{B} \quad N\cdot m$$

→ A small filamentary current loop carrying a current $I$ or a permanent bar magnet of length $l$ is called a magnetic dipole.

## MAGNETIC BOUNDARY CONDITIONS:-

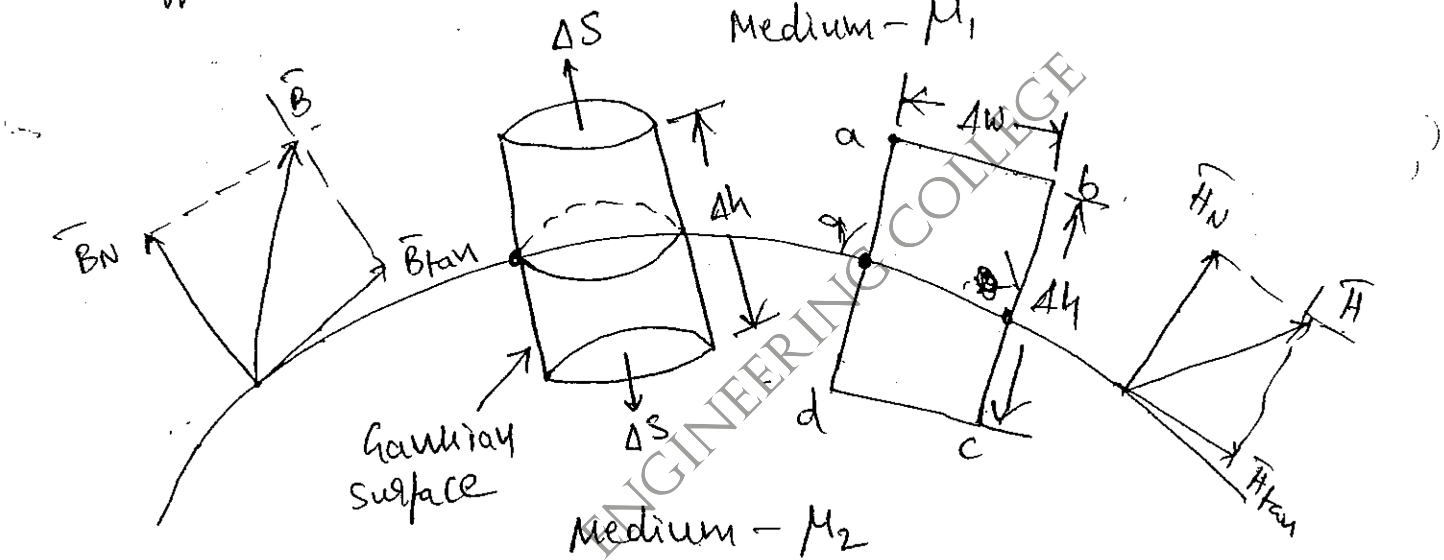
The conditions of the magnetic field existing at the boundary of the two media when the magnetic field passes from one medium to other are called boundary conditions for magnetic fields.

$\bar{B}$ & $\bar{H}$ are resolved into two components

(1) tangential

(2) perpendicular or normal

consider a boundary b/w two homogeneous linear materials with different permeabilities $\mu_1$ & $\mu_2$ as shown in fig.






## BOUNDARY CONDITIONS FOR NORMAL COMPONENT:-

consider a closed Gaussian surface in the form of a right circular cylinder, let the height be $\Delta h$, placed in such a way that $\Delta h/2$ is in medium-1 & $\Delta h/2$ in med-2.

according to Gauss's law for the magnetic field (or) Maxwell's IVth equation.

$$\oint_S \bar{B} \cdot d\bar{S} = 0$$

$$\therefore \oint_{top} \bar{B} \cdot d\bar{S} + \oint_{bottom} \bar{B} \cdot d\bar{S} + \oint_{lateral} \bar{B} \cdot d\bar{S} = 0$$

as $\Delta h \to 0$ the cylinder tends to boundary and only top & bottom surfaces contribute in the surface integral.

Let $\bar{B}$ be $B_{N_1}$ & $B_{N_2}$ in med-1 & 2. as both the surfaces are very small, $B_{N_1}$ & $B_{N_2}$ are const. over their surface.

(11)

$$\oint_{top} \bar{B}\cdot d\bar{s} = B_{N_1} \oint_{top} d\bar{s} = B_{N_1} \Delta S$$

$$\oint_{bottom} \bar{B}\cdot d\bar{s} = B_{N_2} \oint_{bottom} d\bar{s} = -B_{N_2} \Delta S$$

$$\oint_{lateral} \bar{B}\cdot d\bar{s} = 0$$

$\therefore$ $B_{N_1}\Delta S - B_{N_2}\Delta S = 0$ ($\because$ entering & leaving.

$\therefore$ $B_{N_1}\Delta S = B_{N_2}\Delta S$

$$\boxed{B_{N_1} = B_{N_2}}$$

$\therefore$ Thus normal component of $\bar{B}$ is continuous at the boundary

$$\bar{B} = \mu \bar{H}$$

$$\mu_1 H_{N_1} = \mu_2 H_{N_2}$$

$$\boxed{\frac{H_{N_1}}{H_{N_2}} = \frac{\mu_2}{\mu_1} = \frac{\mu_{r_2}}{\mu_{r_1}}}$$

$\therefore$ The normal component of $\bar{H}$ is not continuous at the boundary. the field strength in two media are inversely proportional to their relative permeabilities.

BOUNDARY CONDITIONS FOR TANGENTIAL COMPONENT:-

according to ampere's circuital law,

$$\oint \bar{H}\cdot d\bar{l} = I$$

Consider a rectangular closed path abcda

$$\oint \bar{H}\cdot d\bar{l} = \int_a^b + \int_b^1 + \int_1^c + \int_c^d + \int_d^2 + \int_2^a = I$$

for a small width $\Delta w$ → $H_{tan1}$ in med-1

→ $H_{tan2}$ " med-2

for a small height $\frac{\Delta h}{2}$ → $H_{N_1}$ in med-1

→ $H_{N_2}$ in med-2

assume $\bar{K}$ is the surface current normal to path

$$K \cdot dw = H_{tan1}(\Delta w) + H_{N_1}\left(\frac{\Delta h}{2}\right) + H_{N_2}\left(\frac{\Delta h}{2}\right) - H_{tan2}(\Delta w)$$
$$- H_{N_2}\left(\frac{\Delta h}{2}\right) - H_{N_1}\left(\frac{\Delta h}{2}\right)$$

to get boundary conditions, $\Delta h \to 0$

$$K \cdot dw = H_{tan1}(\Delta w) - H_{tan2}(\Delta w)$$

$$H_{tan1} - H_{tan2} = K$$

in vector form

$$H_{tan1} - H_{tan2} = \bar{a}_{N12} \times \bar{K}$$

$$\therefore \frac{B_{tan1}}{\mu_1} - \frac{B_{tan2}}{\mu_2} = K$$

if the media are not conductors $K = 0$

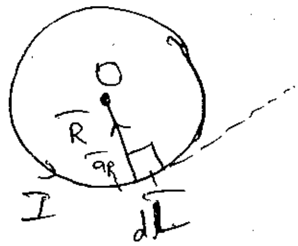
(or) $\boxed{H_{tan1} = H_{tan2}}$

$$\boxed{\frac{B_{tan1}}{B_{tan2}} = \frac{\mu_1}{\mu_2} = \frac{\mu_{r1}}{\mu_{r2}}}$$

A wire of length L is formed into (a) circle, (b) Equilateral $\Delta^{le}$ and (c) square. for the same current I, find the magnetic field $\bar{H}$ at the centre of each.

(28)

∴ (a) wire of length L is formed into circle.




$L = 2\pi R$

$R = \frac{L}{2\pi} = 0.1591\,L$

$d\bar{L}$ is tangential to the circle hence is $\perp^r$ to $\bar{R}$

Biot - Savart law

$$d\bar{H} = \frac{I\,d\bar{L} \times \bar{a}_R}{4\pi R^2}$$

$$d\bar{L} \times \bar{a}_R = |d\bar{L}|\,|\bar{a}_R| \sin\theta\, \bar{a}_N$$

$\bar{a}_N$ = Unit vector normal to the plane containing $d\bar{L}$ & $\bar{a}_R$

$\sin\theta = 1$ at $\theta = 90°$ ∵ angle b/w $d\bar{L}$ & $\bar{a}_R$

$|\bar{a}_R| = 1$ & $|d\bar{L}| = dL$

$$d\bar{H} = \frac{I\,d\bar{L}\,\bar{a}_N}{4\pi R^2}$$

$$\bar{H} = \int \frac{I\,d\bar{L}\,\bar{a}_N}{4\pi R^2}$$

$$\int dL = 2\pi R$$

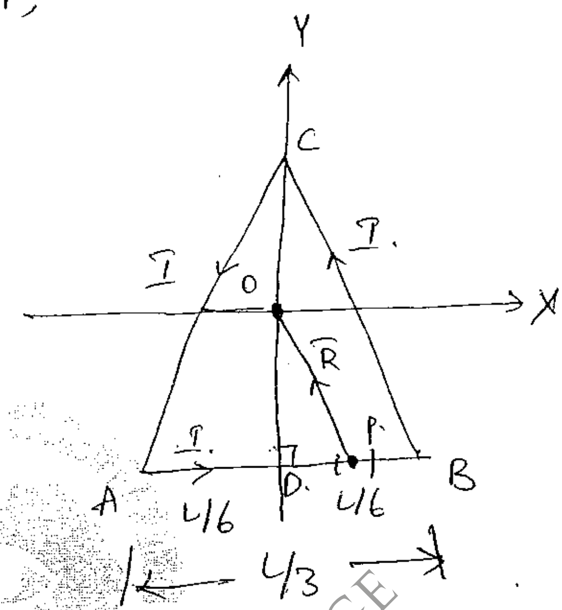
$$\bar{H} = \frac{I \times 2\pi R \times \bar{a}_N}{4\pi R^2}$$

$$= \frac{I}{2R}\,\bar{a}_N$$

$$= \frac{I}{0.3182\,L}\,\bar{a}_N \text{ A/m}$$

b) L is formed into equilateral $\Delta^{e}$, is placed in X-Y plane such that its centre is at the origin,

consider differential length dL at point P, which is at a distance x from D




$l(AC) = \frac{L}{3}$

$l(AD) = \frac{L}{6}$

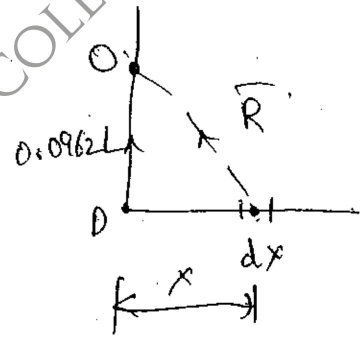
$l(CD) = \sqrt{AC^2 - AD^2}$

$= 0.2886\,L$

$l(OD) = \frac{1}{3}\, l(CD)$

$= \frac{1}{3} \times 0.2886\,L$

$= 0.0962\,L$




$\bar{R} = -x\,\bar{a}_x + 0.0962L\,\bar{a}_y$

$|\bar{R}| = \sqrt{x^2 + (0.0962L)^2}$

$\bar{a}_R = \dfrac{-x\,\bar{a}_x + 0.0962L\,\bar{a}_y}{\sqrt{x^2 + (0.09262L)^2}}$

$d\bar{L} = dx\,\bar{a}_x$

$$d\bar{L} \times \bar{a}_R = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ dx & 0 & 0 \\ -x & 0.0962L & 0 \end{vmatrix} = \frac{0.0962L\,dx\,\bar{a}_z}{\sqrt{x^2 + (0.0962L)^2}}$$

$$d\bar{H} = \frac{I\,d\bar{L} \times \bar{a}_R}{4\pi R^2} = \frac{I\,0.0962L\,dx\,\bar{a}_z}{4\pi\left(x^2 + (0.0962L)^2\right)^{3/2}}$$

$$\bar{H} = \int_{x=-L/6}^{x=L/6} d\bar{H} = 2\int_{x=0}^{L/6} d\bar{H} = \frac{2I\,0.0962L}{4\pi}\int_{x=0}^{L/6} \frac{dx\,\bar{a}_z}{\left[x^2 + (0.0962L)^2\right]^{3/2}}$$

Given $\bar{H} = \frac{y^2 z}{x}\bar{a}_x + \frac{0.5 y^2 z^2}{x^2}\bar{a}_z$ A/m, find the current crossing the square surface at $y=2$ bounded by $x=z=1$ & $x=z=2$

$\therefore \quad d\bar{S} = dx\,dz\,\bar{a}_y$

$$\int_S (\nabla\times\bar{H})\cdot d\bar{S} = I$$

$$\nabla\times\bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \partial/\partial z \\ \frac{y^2 z}{x} & 0 & \frac{y^2 z^2}{2x^2} \end{vmatrix}$$

taking only y component

$$(\nabla\times\bar{H})_y = \left[-\frac{\partial}{\partial x}\left(\frac{y^2 z^2}{2x^2}\right) + \frac{\partial}{\partial z}\left(\frac{y^2 z}{x}\right)\right]\bar{a}_y$$

$$= \left(-\frac{y^2 z^2}{\cancel{2}}\left(\frac{-\cancel{2}}{x^3}\right) + \frac{y^2}{x}\right)\bar{a}_y$$

$$= \left(\frac{y^2 z^2}{x^3} + \frac{y^2}{x}\right)\bar{a}_y$$

$$I = \int_1^2\int_1^2 \left(\frac{y^2 z^2}{x^3} + \frac{y^2}{x}\right)\bar{a}_y \cdot dx\,dz\,\bar{a}_y = \int_1^2\int_1^2 \left(\frac{y^2 z^2}{x^3} + \frac{y^2}{x}\right) dx\,dz$$

at $y=2$

$$I = \int_1^2\int_1^2 \left(\frac{4z^2}{x^3} + \frac{4}{x}\right) dx\,dz$$

$$= 6.27\ \text{A}.$$

A rectangular coil carrying a current of 5A is placed in the magnetic field $\bar{B} = 0.3(\bar{a}_x + \bar{a}_y)T$. The coil is lying in the y-z plane & has dimensions 0.8m × 0.4m find the torque on the coil.

$I = 5$

$B = 0.3(\bar{a}_x + \bar{a}_y)$

$\bar{S} = 0.8 \times 0.4\,\bar{a}_x$

$= 0.32\,\bar{a}_x$

$T = \bar{m} \times \bar{B}$

$= I\bar{S} \times \bar{B}$

$= 5(0.32)\bar{a}_x \times (0.3)(\bar{a}_x + \bar{a}_y) = 5(0.35)(0.3)\bar{a}_z = 0.48\,\bar{a}_z$ N-m

) The current in a coil is increased from 0 to 10A at a uniform rate in 5 sec. It is found that this coil develops self-induced emf of 100V whereas an emf of 20V is induced in a neighbour coil. Compute self-inductance of the first coil and mutual inductance b/w the 2 coils.

$i_1 = 0A$

$i_2 = 10A$

$dt = 5\,sec$

$V_s = 100V$

$V_M = 20V$

(1) $V_s = L\dfrac{di}{dt}$

$di = 10 - 0 = 10A$

$\dfrac{di}{dt} = \dfrac{10}{5} = 2$

$V_s = L(2)$ (or) $L = \dfrac{V_s}{2} = \dfrac{100}{2} = 50\,H$

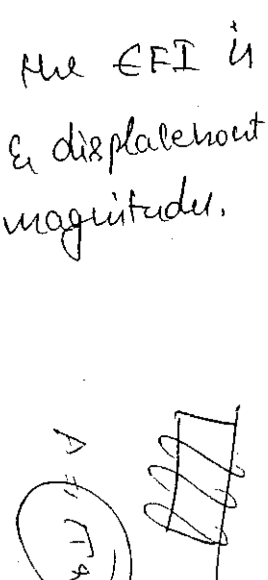
(2) $V_M = M\dfrac{di}{dt} = M(2) \Rightarrow M = \dfrac{20}{2} = 10\,H.$

In a material for which $\sigma = 5.0\,S/m$ & $\varepsilon_r = 1$, the EFI is $E = 250 \sin 10^{10} t$ V/m. find the conduction & displacement current densities and the freq at which they have equal magnitudes.

$\sigma = 5\,S/m$

$\varepsilon_r = 1$

$E = 250 \sin(10^{10} t)$ V/m



(1) In the region $0 < \rho < 0.5m$ in cylindrical coordinates, the current density is $\bar{J} = 4.5 e^{-2\rho} \bar{a_z}$ A/m² and $\bar{J} = 0$ elsewhere. Use ampere's circuital law to find $\bar{H}$.

Sol:
$$I = \oint \bar{J} \cdot \bar{ds}$$
$$\bar{ds} = \rho \, d\rho \, d\phi \, \bar{a_z} \quad (\because \bar{J} \text{ is in } \bar{a_z})$$
$$I = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{0.5} 4.5 e^{-2\rho} \bar{a_z} \cdot \rho \, d\rho \, d\phi \, \bar{a_z}$$
$$I = 1.8676 A$$

Consider a closed path $\rho \geq 0.5m$ that encloses $I = 1.8676 A$

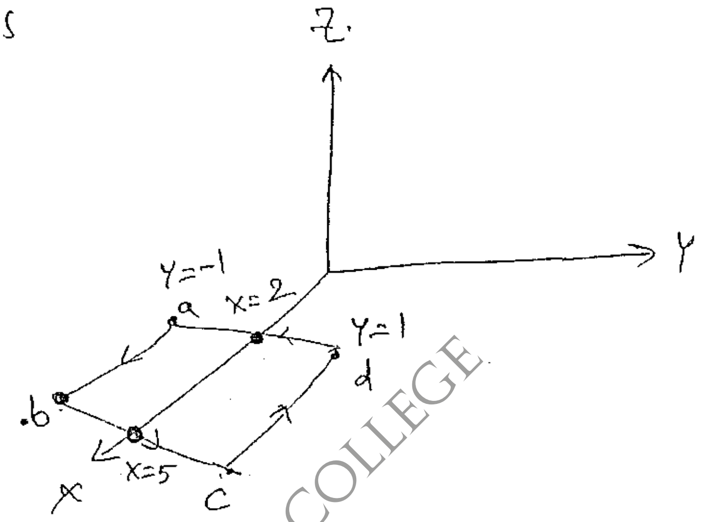
$$\oint \bar{H} \cdot \bar{dL} = I$$
$$\bar{H} = H_\phi \bar{a_\phi}$$
$$\bar{dL} = \rho \, d\phi \, \bar{a_\phi}$$
$$\int_{\phi=0}^{2\pi} H_\phi \rho \, d\phi = I$$
$$H_\phi = \frac{1.8676}{2\pi\rho}$$
$$\bar{H} = \frac{0.2972}{\rho} \bar{a_\phi} \text{ A/m for } \rho \geq 0.5$$

(2) $\bar{H}$ due to a current source is given by $\bar{H} = [y\cos(\alpha x)] \bar{a_x} + (y + e^x) \bar{a_z}$. Describe current density over the yz plane.

Sol:
$$\nabla \times \bar{H} = \bar{J} \Rightarrow \begin{vmatrix} \bar{a_x} & \bar{a_y} & \bar{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\cos(\alpha x) & 0 & y + e^x \end{vmatrix}$$
$$= \left(\frac{\partial}{\partial y}(y + e^x)\right) \bar{a_x} + \left[\frac{\partial y\cos(\alpha x)}{\partial z} - \frac{\partial (y + e^x)}{\partial x}\right] \bar{a_y} + \left[-\frac{\partial}{\partial y} y\cos(\alpha x)\right] \bar{a_z}$$
$$= \bar{a_x} + (0 - e^x) \bar{a_y} + (-\cos\alpha x) \bar{a_z} \Rightarrow \text{on y-z plane } x = 0$$
$$\bar{J} = \bar{a_x} - e^0 \bar{a_y} - \cos 0 \, \bar{a_z} = \bar{a_x} - \bar{a_y} - \bar{a_z} \text{ A/m}^2$$

(5) Evaluate both sides of the Stoke's theorem for the field $\bar{H} = 6xy\,\bar{a}_x - 3y^2\,\bar{a}_y$ A/m and the rectangular path around the region, $2 \le x \le 5$, $-1 \le y \le 1$, $z = 0$. Let the positive direction of $d\bar{s}$ be $\bar{a}_z$.

Sol: $\oint_L \bar{H} \cdot d\bar{l} = \int_S (\nabla \times \bar{H})\, d\bar{s}$




LHS:

$\oint \bar{H} \cdot d\bar{l} = \int_{ab} + \int_{bc} + \int_{cd} + \int_{da} \bar{H} \cdot d\bar{l}$

$\int_{ab} \bar{H} \cdot d\bar{l} = \int_{x=2}^{5} (6xy\,\bar{a}_x - 3y^2\,\bar{a}_y)\, dx\,\bar{a}_x$

$= 63y$

For path ab, $y = -1 = -63$

$\int_{bc} \bar{H} \cdot d\bar{l} = \int_{y=-1}^{1} -3y^2\, dy = -2.$

$\int_{cd} \bar{H} \cdot d\bar{l} = \int_{x=5}^{2} 6xy\, dx = -63y$ (for path cd, $y = 1 \Rightarrow -63$)

$= -63$

$\int_{da} \bar{H} \cdot d\bar{l} = \int_{y=1}^{-1} -3y^2\, dy = 2$

$\therefore \oint \bar{H} \cdot d\bar{l} = -63 - 2 - 63 + 2 = -126$ A.

RHS:

$$\nabla \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy & -3y^2 & 0 \end{vmatrix} = \bar{a}_x(0-0) + \bar{a}_y(0-0) + \bar{a}_z(0-6x) = -6x\,\bar{a}_z$$

$$\int_S (\nabla \times \bar{H}) \cdot d\bar{s} = \int_S (-6x\,\bar{a}_z) \cdot (dx\,dy\,\bar{a}_z) = \int_{y=-1}^{1} \int_{x=2}^{5} -6x\,dx\,dy = \int_{y=-1}^{1} -\frac{3}{\cancel{6}} \frac{x^2}{\cancel{2}}\Big]_2^5 dy$$

$d\bar{s} = dx\,dy\,\bar{a}_z$

$= \int_{-1}^{1} -3(25-4)\, dy$

$= (-63)(1+1)$

$= -126$

$= -126$ A.

(4) $\bar{B} = \left(\frac{2}{\rho}\right) \bar{a}_\phi$ Tesla. Determine the magnetic flux $\phi$ crossing the plane surface defined by $0.5 \le \rho \le 2.5m$ & $0 \le z \le 2m$.

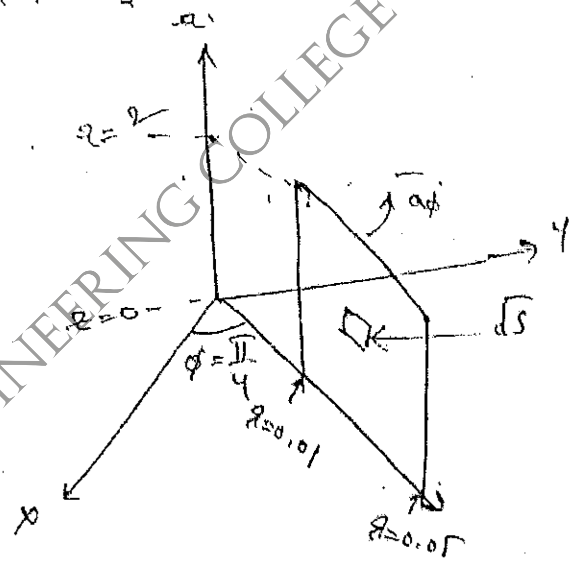
Sol:

$\phi = \int_s \bar{B} \cdot \bar{ds}$ $\Rightarrow \bar{ds} = d\rho\, dz\, \bar{a}_\phi$

$$\phi = \int_{z=0}^{2} \int_{\rho=0.5}^{2.5} \frac{2.0}{\rho}\, d\rho\, dz = 6.4377 \text{ wb}$$

(5) Find the flux passing the portion of the plane $\phi = \pi/4$ defined by $0.01 < \rho < 0.05m$ and $0 < z < 2m$. A current filament of 2.5A is along the z-axis in the $\bar{a}_z$ direction, in free space

Sol:




$$\bar{H} = \frac{I}{2\pi\rho} \bar{a}_\phi$$

$$= \frac{2.5}{2\pi\rho} \bar{a}_\phi$$

$$= \frac{0.3978}{\rho} \bar{a}_\phi$$

$$\bar{B} = \mu_0 \mu_r \bar{H}$$

$$= \frac{4\pi \times 10^{-7} \times 0.3978}{\rho} \bar{a}_\phi$$

$$= \frac{5 \times 10^{-7}}{\rho} \bar{a}_\phi \text{ wb/m}^2$$

$\phi = \int_s \bar{B} \cdot \bar{ds}$ $\quad \bar{ds} = d\rho\, dz\, \bar{a}_\phi$

$$\phi = \int_{z=0}^{2} \int_{\rho=0.01}^{0.05} \frac{5 \times 10^{-7}}{\rho} \bar{a}_\phi\, d\rho\, dz\, \bar{a}_\phi$$

$$= 5 \times 10^{-7} \ln\left[\frac{0.05}{0.01}\right](2) = 1.6094\ \mu\text{wb}$$

scalar magnetic potential: ($V_m$)

If $V_m$ is the scalar magnetic potential then it must satisfy

$$\nabla \times \nabla V_m = 0$$

but $\bar{H} = -\nabla V_m$

$$\nabla \times (-\bar{H}) = 0$$

$$\nabla \times \bar{H} = 0 \; ; \; \nabla \times \bar{H} = \bar{J}$$

$$\bar{J} = 0$$

$$\boxed{\bar{H} = -\nabla V_m} \text{ only for } \bar{J} = 0$$

$$V_{m\,a,b} = -\int_b^a \bar{H} \cdot d\bar{L}$$

$\bar{E} = -\nabla V$

$\bar{H} = -\nabla V_m$

$V_m$ - scalar magnetic potential

$\bar{A}$ - vector magnetic potential

$\nabla \times \nabla V = 0$ $V$ = scalar

$\nabla \cdot (\nabla \times \bar{A}) = 0$ $\bar{A}$ = vector

every scalar & vector $\bar{A}$ must satisfy these identies

Vector magnetic potential ($\bar{A}$):

$$\nabla \cdot (\nabla \times \bar{A}) = 0$$

$$\nabla \cdot \bar{B} = 0$$

$$\bar{B} = \nabla \times \bar{A}$$

$$\nabla \times \bar{H} = \bar{J}$$

$$\nabla \times \frac{\bar{B}}{\mu_0} = \bar{J}$$

$$\nabla \times \bar{B} = \mu_0 \bar{J}$$

$$\nabla \times \nabla \times \bar{A} = \mu_0 \bar{J} \quad (\bar{B} = \nabla \times \bar{A})$$

using vector identity.

$$\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A} = \mu_0 \bar{J}$$

$$\left| \bar{J} = \frac{1}{\mu_0}\left[\nabla \times \nabla \times \bar{A}\right] \right|$$

i) obtain an expression for magnetic vector potential in the region surrounding an infinitely long straight filamentary current I. (17)

sol:- $\bar{H} = \frac{I}{2\pi r} \bar{a}_\phi$

$\bar{B} = \mu_0 \bar{H}$

$\bar{B} = \frac{\mu_0 I}{2\pi r} \bar{a}_\phi$

$\bar{B} = \nabla \times \bar{A}$

$$\frac{\mu_0 I}{2\pi r} \bar{a}_\phi = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\bar{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\bar{a}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right]\bar{a}_z$$

$$\bar{B} = \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} = \frac{\mu_0 I}{2\pi r}$$

As $\bar{B}$ is function of $\rho$ only, the $\bar{A}$ will also change with $\rho$ only & will be const. w.r.t. z.

$$\frac{\partial A_\rho}{\partial z} = 0$$

$$\frac{\partial A_z}{\partial \rho} = -\frac{\mu_0 I}{2\pi \rho}$$

$$A_z = -\int \frac{\mu_0 I}{2\pi \rho} d\rho + c_1$$

$$= -\frac{\mu_0 I}{2\pi \rho} \ln(\rho) + c_1$$

to find c, let us find ref zero where $A_z$ will be zero. let $A_z = 0$ at $\rho = \rho_0$

$$0 = -\frac{\mu_0 I}{2\pi \rho} \ln[\rho_0] + c_1$$

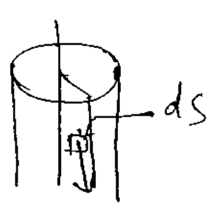
$$c_1 = \frac{\mu_0}{2\pi \rho} \ln[\rho_0]$$

$$A_z = -\frac{\mu_0 I}{2\pi} \ln(\rho) + \frac{\mu_0 I}{2\pi} \ln[\rho_0]$$

$$A_z = \frac{\mu_0 I}{2\pi} \ln\left(\frac{\rho_0}{\rho}\right) \qquad \bar{A} = A_z \bar{a}_z = \frac{\mu_0 I}{2\pi} \ln\left[\frac{\rho_0}{\rho}\right] \bar{a}_z \text{ Wb/m}$$

2) A radial field, $\bar{H} = \frac{2.39 \times 10^6}{r} \cos\phi \, \bar{a}_r$ A/m exists in free space. Find the magnetic flux crossing the surface defined by $0 \le \phi \le \pi/4$ and $0 \le z \le 1$ m.

Sol:




$$\phi = \int_S \bar{B} \cdot \bar{ds}$$

$$\bar{ds} = \rho \, d\phi \, dz \, \bar{a}_\rho$$

$$\therefore \phi = \int_S \mu_0 \bar{H} \cdot \bar{ds}$$

$$= \mu_0 \int_S \frac{2.39 \times 10^6}{\rho} \cos\phi \, \bar{a}_\rho \cdot \rho \, d\phi \, dz \, \bar{a}_\rho$$

$$= \mu_0 \int_{z=0}^{1} \int_{\phi=0}^{\pi/4} 2.39 \times 10^6 \cos\phi \, d\phi \, dz$$

$$= 2.39 \times 10^6 \mu_0 \left[\sin\phi\right]_0^{\pi/4} \left[z\right]_0^1$$

$$\phi = 2.39 \times 10^6 \times 4\pi \times 10^{-7} \times \left[\sin\frac{\pi}{4} - \sin 0\right][1-0]$$

$$= 2.1236 \text{ wb}$$

3) Find the flux passing the portion of the plane $\phi = \pi/4$ defined by $0.01 < r < 0.05$ m & $0 < z < 2$ m. A current filament of 2.5 A is along z axis in the $\bar{a}_z$ direction in free space.

$$\bar{H} = \frac{I}{2\pi\rho} \bar{a}_\phi$$

$$= \frac{2.5}{2\pi\rho} \bar{a}_\phi$$

$$= \frac{0.3978}{\rho} \bar{a}_\phi$$

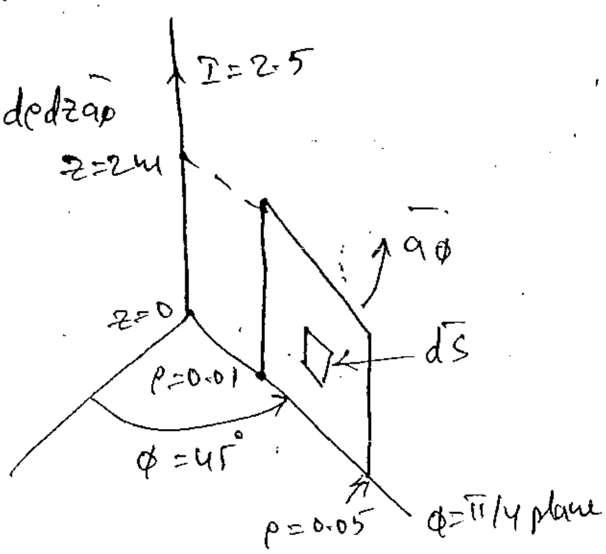
$$\bar{B} = \frac{5 \times 10^{-7}}{\rho} \bar{a}_\phi$$

$$\phi = \int \bar{B} \cdot \bar{ds}$$

$$\bar{ds} = d\rho \, dz \, \bar{a}_\phi$$

$$\phi = \int_{z=0}^{2} \int_{\rho=0.01}^{0.05} \frac{5 \times 10^{-7}}{\rho} \bar{a}_\phi \cdot d\rho \, dz \, \bar{a}_\phi$$

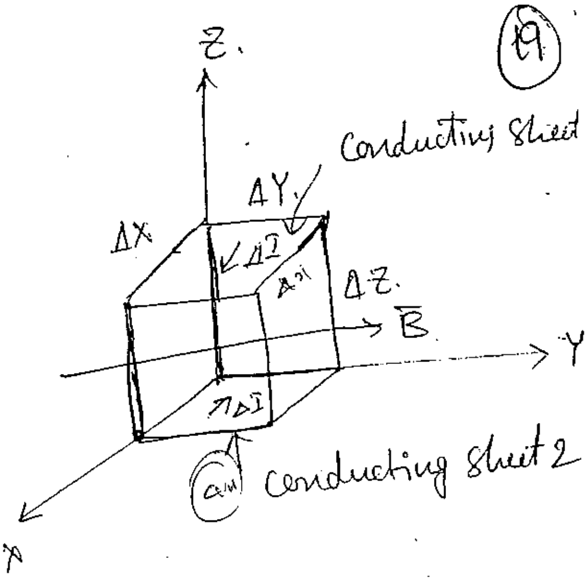
$$\phi = 1.6094 \, \mu\text{Wb}$$



## ENERGY STORED IN A MAGNETIC FIELD:-

inductor energy is stored in magnetic field.

$$\therefore W_m = \frac{1}{2} L I^2$$




Consider a differential volume in a magnetic field $\bar{B}$ as shown in fig.

Consider that at the top & bottom surfaces of a differential volume, conducting sheets with $\Delta I$ are present

$$\Delta L = \frac{\Delta \phi}{\Delta I} = \frac{B \Delta S}{\Delta I} \qquad \left[\because \phi = \int_S \bar{B} \cdot d\bar{S}\right.$$

$\Delta S = \Delta x \Delta z$ = differential surface area.

$$\Delta L = \frac{B(\Delta x \Delta z)}{\Delta I}$$

$$B = \mu H$$

$$\Delta L = \frac{\mu H \Delta x \Delta z}{\Delta I}$$

$\Delta I = H(\Delta Y)$ — the current flowing through the conducting sheets present at the top & bottom, is in Y direction

$\Delta W_m = \frac{1}{2} \Delta L \Delta I^2$ — energy stored in the inductance differential volume

$$\Delta W_m = \frac{1}{2}\left[\frac{\mu H \Delta x \Delta z}{H \Delta Y}\right][H \Delta Y]^2$$

$$\Delta W_m = \frac{1}{2}\mu H^2 (\Delta x \Delta Y \Delta z)$$

$$\boxed{\Delta W_m = \frac{1}{2}\mu H^2 \Delta V}$$

The magnetostatic energy density function is defined as

$$w_m = \lim_{\Delta V \to 0} \frac{\Delta W_m}{\Delta V} = \frac{1}{2}\mu H^2$$

different forms

$$w_m = \frac{1}{2}(\mu H)H = \frac{1}{2} BH$$

$$= \frac{1}{2} B\left(\frac{B}{\mu}\right) = \frac{B^2}{2\mu}$$

In a linear medium,

$$W_m = \int w_m \, dV$$

$$W_m = \frac{1}{2}\int \bar{B} \cdot \bar{H} \, dV$$

$$= \frac{1}{2}\int \mu H^2 \, dV$$

$$= \frac{1}{2}\int \frac{B^2}{\mu} \, dV$$

# UNIT - IV
## MAGNETIC POTENTIAL

## 7.13 Magnetic Scalar and Vector Potentials

In electrostatics, it is seen that there exists a scalar electric potential V which is related to the electric field intensity $\bar{E}$ as $\bar{E} = -\nabla V$.

Is there any scalar potential in magnetostatics related to magnetic field intensity $\bar{H}$ ?

In case of magnetic fields there are two types of potentials which can be defined :

1. The scalar magnetic potential denoted as $V_m$.

2. The vector magnetic potential denoted as $\bar{A}$.

To define scalar and vector magnetic potentials, let us use two vector identities which are listed as the properties of curl, earlier.

$$\nabla \times \nabla V = 0, \qquad V = \text{Scalar} \qquad \dots (1)$$

$$\nabla \cdot (\nabla \times \bar{A}) = 0, \qquad \bar{A} = \text{Vector} \qquad \dots (2)$$

Every Scalar V and Vector $\bar{A}$ must satisfy these identities.

### 7.13.1 Scalar Magnetic Potential

If $V_m$ is the scalar magnetic potential then it must satisfy the equation (1),

$$\therefore \qquad \nabla \times \nabla V_m = 0 \qquad \dots (3)$$

But the scalar magnetic potential is related to the magnetic field intensity $\bar{H}$ as,

$$\bar{H} = -\nabla V_m \qquad \dots (4)$$

Using in equation (3),

$$\therefore \qquad \nabla \times (-\bar{H}) = 0 \qquad \text{i.e.} \quad \nabla \times \bar{H} = 0 \qquad \dots (5)$$

$$\text{But} \qquad \nabla \times \bar{H} = \bar{J} \qquad \text{i.e.} \quad \bar{J} = 0 \qquad \dots (6)$$

Thus scalar magnetic potential $V_m$ can be defined for source free region where $\bar{J}$ i.e. current density is zero.

$$\therefore \quad \bar{H} = -\nabla V_m \quad \text{only for } \bar{J} = 0 \quad \text{... (7)}$$

Similar to the relation between $\bar{E}$ and electric scalar potential, magnetic scalar potential can be expressed interms of $\bar{H}$ as,

$$\boxed{V_{m\,a,b} = -\int_b^a \bar{H} \cdot d\bar{L} \quad \text{... specified path}}$$

### 7.13.2 Laplace's Equation for Scalar Magnetic Potential

It is known that as monopole of magnetic field is non existing,

$$\oint \bar{B} \cdot d\bar{S} = 0 \quad \text{... (8)}$$

Using Divergence theorem,

$$\oint \bar{B} \cdot d\bar{S} = \int_{vol} (\nabla \cdot \bar{B})\, dv = 0 \quad \text{... (9)}$$

$$\therefore \quad \nabla \cdot \bar{B} = 0 \quad \text{... (10)}$$

$$\therefore \quad \nabla \cdot (\mu_0 \bar{H}) = 0 \quad \text{but } \mu_0 \neq 0 \quad \text{... (11)}$$

$$\therefore \quad \nabla \cdot \bar{H} = 0 \quad \text{... (12)}$$

$$\therefore \quad \nabla \cdot (-\nabla V_m) = 0 \quad \text{... using } \bar{H} = -\nabla V_m$$

$$\therefore \quad \boxed{\nabla^2 V_m = 0 \quad \text{for} \quad \bar{J} = 0} \quad \text{... (13)}$$

This is Laplace's equation for scalar magnetic potential. This is similar to the Laplace's equation for scalar electric potential $\nabla^2 V = 0$.

### 7.13.3 Vector Magnetic Potential

The vector magnetic potential is denoted as $\bar{A}$ and measured in Wb/m. It has to satisfy equation (2) that divergence of a curl of a vector is always zero.

$$\therefore \quad \nabla \cdot (\nabla \times \bar{A}) = 0 \quad \text{... } \bar{A} = \text{Vector magnetic potential}$$

$$\text{But} \quad \nabla \cdot \bar{B} = 0 \quad \text{... From equation (10)}$$

$$\therefore \quad \boxed{\bar{B} = \nabla \times \bar{A}} \quad \text{... (14)}$$

Thus curl of vector magnetic potential is the flux density.

$$\text{Now} \quad \nabla \times \bar{H} = \bar{J}$$

$$\therefore \quad \nabla \times \frac{\bar{B}}{\mu_0} = \bar{J} \quad \text{... } \bar{B} = \mu_0 \bar{H}$$

$$\therefore \qquad \nabla \times \bar{B} = \mu_0 \bar{J} \qquad \ldots \; \bar{B} = \nabla \times \bar{A}$$

$$\therefore \qquad \nabla \times \nabla \times A = \mu_0 J \qquad \ldots (15)$$

Using vector identity to express left hand side we can write,

$$\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A} = \mu_0 \bar{J}$$

$$\therefore \qquad \bar{J} = \frac{1}{\mu_0}\left[\nabla \times \nabla \times \bar{A}\right] = \frac{1}{\mu_0}\left[\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}\right] \qquad \ldots (16)$$

Thus if vector magnetic potential is known then current density $\bar{J}$ can be obtained. For defining $\bar{A}$ the current density need not be zero.

### 7.13.4 Poisson's Equation for Magnetic Field

In a vector algebra, a vector can be fully defined if its curl and divergence are defined.

For a vector magnetic potential $\bar{A}$, its curl is defined as $\nabla \times \bar{A} = \bar{B}$ which is known.

But to completely define $\bar{A}$ its divergence must be known. Assume that $\nabla \cdot \bar{A}$, the divergence of $\bar{A}$ is zero. This is consistent with some other conditions to be studied later in time varying magnetic fields. Using in equation (16),

$$\bar{J} = \frac{1}{\mu_0}\left[-\nabla^2 \bar{A}\right]$$

$$\therefore \qquad \boxed{\nabla^2 \bar{A} = -\mu_0 \bar{J}} \qquad \ldots (17)$$

This is the Poisson's equation for magnetostatic fields.

### 7.13.5 $\bar{A}$ due to Differential Current Element

Consider the differential element $d\bar{L}$ carrying current I. Then according to Biot-Savart law the vector magnetic potential $\bar{A}$ at a distance R from the differential current element is given by,

$$\boxed{\bar{A} = \oint \frac{\mu_0 I \, d\bar{L}}{4\pi R} \text{ Wb/m}} \qquad \ldots (18)$$

For the distributed current sources, $I d\bar{L}$ can be replaced by $\bar{K}\,dS$ where $\bar{K}$ is surface current density.

$$\therefore \qquad \boxed{\bar{A} = \oint_S \frac{\mu_0 \bar{K}\, dS}{4\pi R} \text{ Wb/m}} \qquad \ldots (19)$$

The line integral becomes a surface integral. If the volume current density $\bar{J}$ is given in $A/m^2$ then $I\, d\bar{L}$ can be replaced by $\bar{J}\, dv$ where dv is differential volume element.

$$\therefore \qquad \boxed{A = \oint_{vol} \frac{\mu_0 \bar{J}\, dv}{4\pi R} \text{ Wb/m}} \qquad \ldots (20)$$

## ...GNETIC DIPOLE MOMENT IN THE MATERIAL:

...ignetic materials are classified on the basis of presence of ...agnetic dipole moments in the materials.

...n general there are 3 important contributions to the angular ...oment of an atom

(i) Orbital magnetic dipole moment

(ii) Electron spin magnetic moment

(iii) Nuclear spin magnetic moment

## ...ASSIFICATION OF MAGNETIC MATERIALS:

...) Diamagnetic — lead, copper, silicon, diamond, graphite, sulphur, sodium chloride & inert gas

...) paramagnetic — Potassium, tungston, $O_2$

...i) ferromagnetic — Iron, nickel and cobalt

...) antiferromagnetic — oxides, chlorides & sulphides

...) ferrimagnetic — nickel ferrite, nickel-zinc-ferrite & iron-oxide-magnetite

...i) ~~....~~ supermagnetic — magnetic tapes used for audio, video & data recordings

...NDUCTANCE: A wire or conductor of certain length, when twisted ...nto coil becomes a basic inductor. ~~..~~ every current carrying conductor produces magnetic field $\bar{B}$ and there exists a self-inductance. When 2 such coils are placed very close to each other, there exists a mutual ~~.~~ inductance b/w the two.

...ELF-INDUCTANCE: When a closed conducting path of a circuit carries current I, a magnetic field $\bar{B}$ is produced. This causes a magnetic flux $\phi$

$$\phi = \int_S \bar{B} \cdot d\bar{S}$$

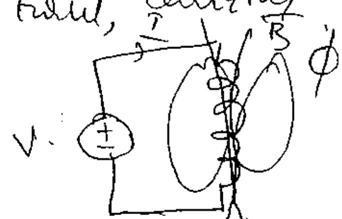
In the ckt or closed path consisting of N turns, the flux produced by the magnetic field $\bar{B}$ links with each turn of the ckt.

The flux linkage is defined as the product of no. of turns N and the total flux $\phi$ linking each of the turn. It is denoted by $\lambda$.

$$\lambda = N\phi \quad wb.t$$

→ The flux linking with each turn of the ckt is proportional to the current flowing through the ckt.

→ The flux linkage with the ckt of N turns, carrying current I is shown in fig.



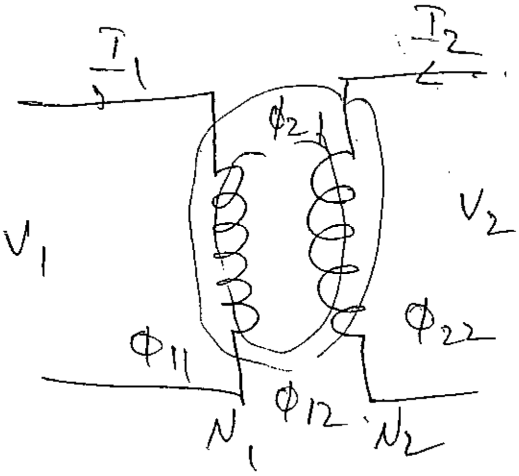

→ The ratio of the total flux linkage to the current flowing through the ckt is called inductance

$$L = \frac{N\phi}{I} = \frac{\lambda}{I} \quad H$$

→ The inductance is also known as self-inductance of the ckt as the flux produced by the current flowing through the ckt links with the same ckt. It is measured in H.

wb-turn/A.

MUTUAL INDUCTANCE:




Consider that 2 different ckts with selfinductances $L_1$ & $L_2$ are kept close to each other

$N_1$ & $N_2$

$I_1$ & $I_2$

$\phi_{11}$, $\phi_{22}$

part of flux $\phi_{11}$ links ckt 2 is by $\phi_{12}$

$\phi_{22}$ " ckt 1 " $\phi_{21}$

$$M_{21} = \frac{N_1\phi_{21}}{I_2} \quad ; \quad M_{12} = \frac{N_2\phi_{12}}{I_1}$$

$$M_{12} = M_{21} = M$$

EFFICIENT OF COUPLING b/w TWO ckt:

$$L_1 = \frac{N_1 \phi_{11}}{I_1} = \frac{\lambda_{11}}{I_1}$$

$$L_2 = \frac{N_2 \phi_{22}}{I_2} = \frac{\lambda_{22}}{I_2}$$

the flux linking with ck 2 due to current in ckt 1 is denoted by $\phi_{12}$. this flux is actually the part of the total flux produced by $I_1$ in ckt 1.

$$\phi_{12} = K_1 \phi_{11}$$

$$\phi_{21} = K_2 \phi_{22}$$

$$M_{12} = \frac{N_1 \phi_{21}}{I_2} = \frac{N_1 (K_2 \phi_{22})}{I_2}$$

$$M_{21} = \frac{N_2 \phi_{12}}{I_1} = \frac{N_2 (K_1 \phi_{11})}{I_1}$$

$$M_{12} = M_{21} = M$$

$$M^2 = M_{12} \cdot M_{21} = \frac{N_1 (K_2 \phi_{22})}{I_2} \cdot \frac{N_2 (K_1 \phi_{11})}{I_1}$$

$$= K_1 \cdot K_2 \left(\frac{N_1 \phi_{11}}{I_1}\right)\left(\frac{N_2 \phi_{22}}{I_2}\right)$$

$$= K_1 K_2 L_1 L_2$$

$$M = \sqrt{K_1 K_2} \cdot \sqrt{L_1 L_2}$$

$$M = K\sqrt{L_1 L_2}$$

$$K = \frac{M}{\sqrt{L_1 L_2}}$$

→ when the 2 magnetic ckts are coupled together in series aiding

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$+$

then $L_{eq} = L_1 + L_2 + 2M$

minus

→ opposing ⇒ $L_{eq} = L_1 + L_2 - 2M$

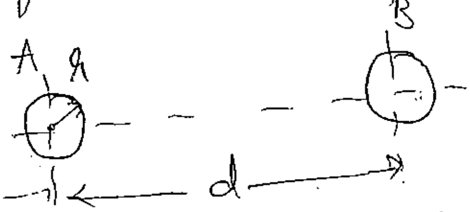
parallel aiding $L_{eq} = \dfrac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$ minus

opposing $L_{eq} = \dfrac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$ $+$

## INDUCTANCE OF A SINGLE PHASE TWO-WIRE LINE:

A 1-ph line consists of 2 parallel conductors which form a rectangular loop of one turn. When an AC flows through such a loop, a changing magnetic flux is set up.

Consider a 1-ph OH line consisting of 2 parallel conductors A and B spaced d m apart. Conductors A & B carry the same amount of current but in opposite direction




$\therefore I_A + I_B = 0$

There will be flux linkages with conductor A due to its own current $I_A$ and also due to mutual inductance effect of current $I_B$ in the conductor B.

⇒ flux linkages with conductor A due to its own current

$$= \frac{\mu_0 I_A}{2\pi}\left[\frac{1}{4} + \int_r^{\infty}\frac{dx}{x}\right] \rightarrow (1) \qquad \psi_{int} + \psi_{ext}$$

⇒ flux linkages with conductor A due to current $I_B$

$$= \frac{\mu_0 I_B}{2\pi}\int_d^{\infty}\frac{dx}{x} \rightarrow (2) \qquad \psi_{ext} \text{ only}$$

total flux linkages with conductor A is

$$\psi_A = (1) + (2)$$

$$= \frac{\mu_0 I_A}{2\pi}\left[\frac{1}{4} + \int_r^{\infty}\frac{dx}{x}\right] + \frac{\mu_0 I_B}{2\pi}\int_d^{\infty}\frac{dx}{x}$$

$$= \frac{\mu_0}{2\pi}\left\{\left(\frac{1}{4} + \int_r^{\infty}\frac{dx}{dx}\right)I_A + I_B\int_d^{\infty}\frac{dx}{x}\right\}$$

$$= \frac{\mu_0}{2\pi}\left[\left(\frac{1}{4} + \log_e \infty - \log_e r\right] I_A + (\log_e \infty - \log_e d] I_B\right]$$

$$= \frac{\mu_0}{2\pi}\left[\frac{I_A}{4} + \log_e \infty (I_A + I_B) - I_A \log_e r - I_B \log_e d\right]$$

$$= \frac{\mu_0}{2\pi}\left[\frac{I_A}{4} - I_A \log_e r - I_B \log_e d\right]$$

$I_A + I_B = 0$

$-I_B = I_A \Rightarrow -I_B \log_e d = I_A \log_e d$

$$\psi_A = \frac{\mu_0}{2\pi}\left[\frac{I_A}{4} + I_A \log_e d - I_A \log_e r\right]$$

$$= \frac{\mu_0}{2\pi}\left[\frac{I_A}{4} + I_A \log_e \frac{d}{r}\right]$$

$$= \frac{\mu_0 I_A}{2\pi}\left[\frac{1}{4} + \log_e \frac{d}{r}\right]$$

$$L_A = \frac{\psi_A}{I_A} = \frac{\mu_0}{2\pi}\left[\frac{1}{4} + \log_e \frac{d}{r}\right]$$

$$L_A = \frac{\cancel{4\pi} \times 10^{-7}}{\cancel{2\pi}}\left[\frac{1}{4} + \log_e \frac{d}{r}\right]$$

$$= 10^{-7}\left[\frac{1}{2} + 2\log_e \frac{d}{r}\right]$$

$$\text{loop inductance} = 2L_A = 10^{-7}\left[1 + 4\log_e \frac{d}{r}\right] H/m$$

$$\boxed{\therefore \text{Loop inductance} = 10^{-7}\left[1 + 4\log_e \frac{d}{r}\right] H/m}$$

$\lambda_1 = \frac{\mu I l}{8\pi}$

$\lambda_2 = \lambda_{2}, r$ $\int$ $\phi = a$

Q What is the max torque on a square loop of 100 turns in a field of uniform flux density 1 $wb/m^2$. The loop has 10cm side & carries a curr of 3A. ~~What is the max torque~~ ~~mag moment of the loop~~.

$\overline{T} = \overline{m} \times \overline{B} = |m||B| \sin\theta$, $\sin\theta = 1$

$T_{max} = mB = ISB$ | $S = (10\,cm)^2 = 10^2 \times 10^{-4}\,m^2$

$= 3\,Nm$

Q If the mag field is $\overline{H} = \frac{0.01}{\mu_0}\hat{a}_x$ A/m. What is the force of a charge of $Q = 1pC$ moving with a velocity of $\overline{v} = 10^6 \hat{a}_y$ m/s $\longrightarrow -10^{-8}\hat{a}_z$ N

Q If a point charge 4C moves with a velocity of $v = 5\hat{a}_x + 6\hat{a}_y - 7\hat{a}_z$ m/s. Find the force exerted, if the flux density is $\overline{B} = 5\hat{a}_x + 7\hat{a}_y + 9\hat{a}_z$ wb/$m^2$

Ans: $412\hat{a}_x - 320\hat{a}_y + 20\hat{a}_z$.

Q. Cal the F on a st cond of length 30cm carrying a curr of 5A along the z axis the mag field is $\overline{B} = 3.5 \times 10^{-3}(\hat{a}_x - \hat{a}_y)$ T.

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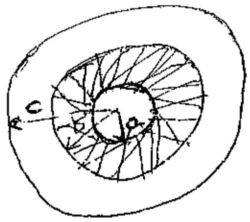
$a < \rho < b$:




$H = \frac{I}{2\pi\rho}\hat{a}_\phi$

$B = \mu H$

$\phi = \int B \cdot dS$ | $dS = d\rho\, dz\, \hat{a}_\phi$

$= \int \frac{\mu I}{2\pi\rho}\hat{a}_\phi \cdot d\rho\, dz\, \hat{a}_\phi$

$= \frac{\mu I}{2\pi}\int_a^b \rho\, d\rho \int_0^d dz$

$= \frac{\mu I}{2\pi}[\ln\rho]_a^b\,[z]_0^d$

$\phi = \frac{\mu I d}{2\pi}\ln\left(\frac{b}{a}\right)$

$L = \frac{\phi}{I} = \frac{\mu d}{2\pi}\ln\left(\frac{b}{a}\right)$ H

Inductance per unit length,

$L/_{unit\ length} = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)$. H/m.

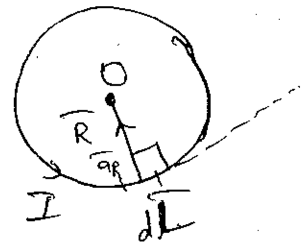
16/10/19 EEE

All present.

A wire of length L is formed into (a) circle, (b) Equilateral $\Delta^{le}$ and (c) square. for the same current I, find the magnetic field $\bar{H}$ at the centre of each.

(26)

∴ (a) wire of length L is formed into circle.




$L = 2\pi R$

$R = \frac{L}{2\pi} = 0.1591 L$

$d\bar{L}$ is tangential to the circle hence is $\perp^r$ to $\bar{R}$

Biot - Savart law

$$d\bar{H} = \frac{I d\bar{L} \times \bar{a}_R}{4\pi R^2}$$

$$d\bar{L} \times \bar{a}_R = |d\bar{L}| |\bar{a}_R| \sin\theta \, \bar{a}_N$$

$\bar{a}_N$ = Unit vector normal to the plane containing $d\bar{L}$ & $\bar{a}_R$

$\sin\theta = 1$ at $\theta = 90°$ ; angle b/w $d\bar{L}$ & $\bar{a}_R$

$|\bar{a}_R| = 1$ & $|d\bar{L}| = dL$

$$d\bar{H} = \frac{I d\bar{L} \bar{a}_N}{4\pi R^2}$$

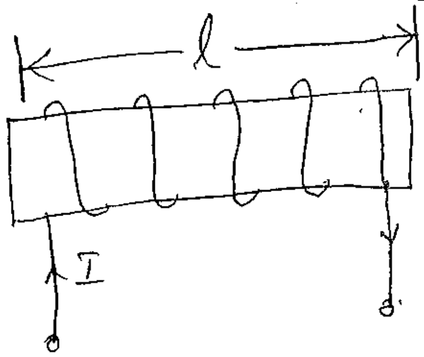
$$\bar{H} = \int \frac{I d\bar{L} \bar{a}_N}{4\pi R^2}$$

$$\int dL = 2\pi R$$

$$\bar{H} = \frac{I \times 2\pi R \times \bar{a}_N}{4\pi R^2}$$

$$= \frac{I}{2R} \bar{a}_N$$

$$= \frac{I}{0.3182 L} \bar{a}_N \text{ A/m}$$




## INDUCTANCE OF A SOLENOID:-

Consider a solenoid of N turns

the current flowing through

solenoid be I amp.

the length of the solenoid be

l and cross-sectional area be A.

The field intensity inside the solenoid is

$$H = \frac{NI}{l} \ A/m$$

total flux linkage is given by,

$$N\phi = N(B)(A)$$

$$= N(\mu H)(A)$$

$$N\phi = \mu N H A$$

$$= \mu N\left(\frac{NI}{l}\right)A$$

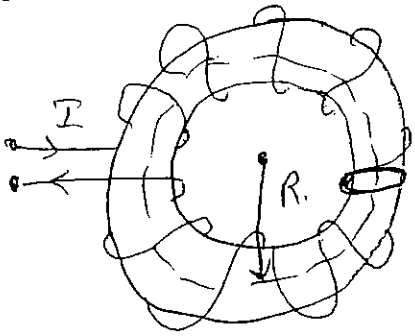
$$= \mu \frac{N^2 I}{l} A$$

$$\therefore \quad L = \frac{\text{Total flux linkage}}{\text{total current}} = \frac{\mu N^2 I A}{l(I)}$$

$$\boxed{L = \frac{\mu N^2 A}{l} \ H}$$

## INDUCTANCE OF A TOROID:-

Consider a toroidal ring with N turns & carrying current I. Let the radius of the troid R as shown in fig.




The magnetic flux density inside a toroidal ring is given by

$$B = \frac{\mu N I}{2\pi R} \qquad \mu H = B \qquad H = \frac{NI}{2\pi R}$$

total flux linkage of a toroidal ring having N turns

$$= N\phi$$

$$= N(B)(A)$$

$$= N\left(\frac{\mu N I}{2\pi R}\right)(A)$$

$$= \frac{\mu N^2 I A}{2\pi R}$$

$$L = \frac{\text{total flux linkage}}{\text{total current}}$$

$$= \frac{\mu N^2 \cancel{I} A}{2\pi R \times \cancel{I}}$$

$$\boxed{L = \frac{\mu N^2 A}{2\pi R}} \qquad A = \pi r^2 \ m^2$$

for a toroid with N no. of turns

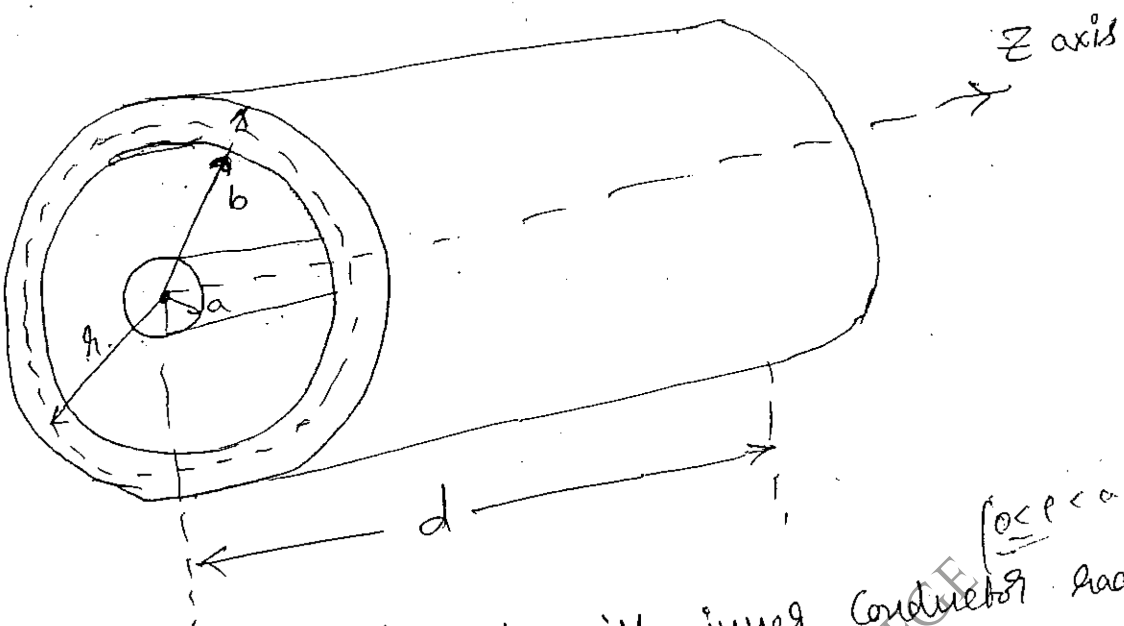
h as the height

$r_1$ inner radius

$r_2$ outer radius

$$\boxed{L = \frac{\mu N^2 h}{2\pi} \ln\left(\frac{r_2}{r_1}\right)}$$

# INDUCTANCE OF A CO-AXIAL CABLE:-




$(0 \le \rho < a)$

Consider a co-axial cable with inner conductor radius 'a' and outer conductor radius 'b' as shown in fig.

Let the current through the co-axial cable be I.

For the cable the field intensity at any point b/w inner and outer conductors is given by

$$H = \frac{I}{2\pi r} \quad \text{where } a < r < b$$

but $B = \mu H = \frac{\mu I}{2\pi r}$

Now assume that the axis of the cable is along z-axis the magnetic flux density will be in radial plane extending from $r = a$ to $r = b$

$z = 0$ to $z = d$

let $\bar{B} = \frac{\mu I}{2\pi r} \bar{a}_\phi$

total magnetic flux, $\phi = \int_s \bar{B} \cdot d\bar{s}$

$d\bar{S} = dr\,dz\,\bar{a}_\phi$

$$\phi = \int_{z=0}^{d}\int_{r=a}^{b} \left(\frac{\mu I}{2\pi r}\bar{a}_\phi\right)\cdot\left(dr\,dz\,\bar{a}_\phi\right)$$

$$= \frac{\mu I}{2\pi}\left[z\right]_0^d \left[\ln r\right]_a^b$$

$$= \frac{\mu I d}{2\pi}\ln\left(\frac{b}{a}\right)$$

$$L = \frac{\text{total flux linkage}}{\text{total current}}$$

$$= \frac{\frac{\mu I d}{2\pi}\ln\left(\frac{b}{a}\right)}{I}$$

$$\boxed{L = \frac{\mu d}{2\pi}\ln\left(\frac{b}{a}\right)\ H}$$

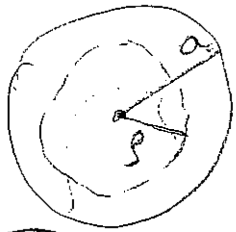
inductance of co-axial cable per unit length is

$$\boxed{L = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)\ H/m.}$$

## FLUX LINKAGES DUE TO A SINGLE CURRENT CARRYING CONDUCTOR:

(17)

Consider a long straight cylindrical conductor of radius 'r' m and carrying a current $I$ amp (rms). This current will setup magnetic field. The magnetic lines of force will exist inside the conductor as well as outside the conductor. Both these fluxes will contribute to the inductance of the conductor.




### FLUX LINKAGES DUE TO INTERNAL FLUX:

we know that $H_x = \frac{I_x}{2\pi x}$

assuming uniform current density

$$I_x = \frac{\pi x^2}{\pi r^2} I$$

$$= \frac{x^2}{r^2} I$$

$$H_x = \frac{x^2}{r^2} \times I \times \frac{1}{2\pi x} = \frac{x}{2\pi r^2} I \ \ AT/m$$

$$\boxed{B_x = \mu_0 \mu_r H_x}$$

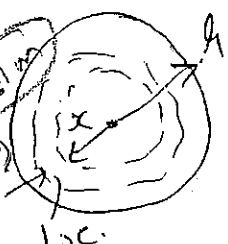
$$= \frac{\mu_0 \mu_r x}{2\pi r^2} I$$

$$= \frac{\mu_0 x I}{2\pi r^2} \ wb/m^2$$

$H_{int} = \frac{I \rho}{2\pi a^2} \Rightarrow \mu$

$d\phi = \frac{\mu_0 I \rho}{2\pi a^2} (d\rho\, dz)$

$dz = 1m$




flux $d\phi$ through a cylindrical shell of radial thickness $dx$ and axial length 1m is given by

$$d\phi = B_x \times 1 \times dx$$

$$= \frac{\mu_0 x I}{2\pi r^2} dx.$$

this flux links with current $I_x$ only.

$\therefore$ flux linkage/m length of the conductor $= d\psi = \frac{\pi x^2}{\pi r^2} d\phi$

$$= \frac{\pi x^2}{\pi r^2} \times \frac{\mu_0 x I}{2\pi r^2} dx$$

$$= \frac{\mu_0 I x^3}{2\pi r^4} dx$$

total flux linkages from origin to conductor surface

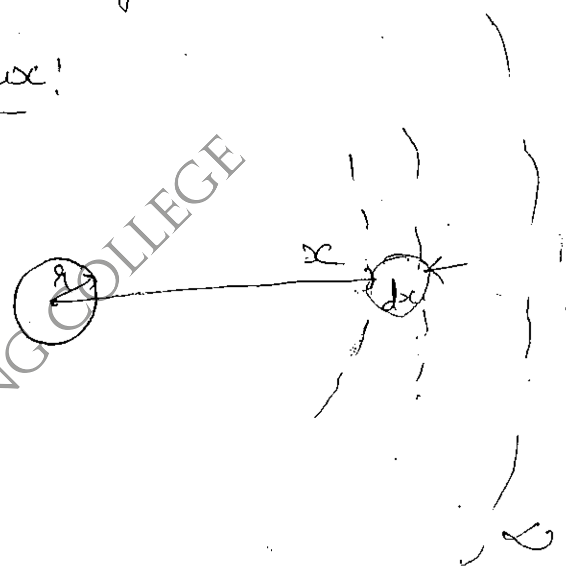
$$\psi_{int} = \int_0^r \frac{\mu_0 I x^3}{2\pi r^4} dx$$

$$= \frac{\mu_0 I}{8\pi} \text{ wb-turn/m length}$$

ii) Flux linkages due to External flux:

$$H_x = \frac{I}{2\pi x}$$

$$B_x = \mu_0 H_x = \frac{\mu_0 I}{2\pi x}$$




$$d\phi = B_x dx = \frac{\mu_0 I}{2\pi x} dx$$

$$d\psi = d\phi = \frac{\mu_0 I}{2\pi x} dx$$

$$\psi_{ext} = \int_r^{\infty} \frac{\mu_0 I}{2\pi x} dx \text{ wb-turn}$$



$\therefore$ total flux linkages, $\psi = \psi_{int} + \psi_{ext}$

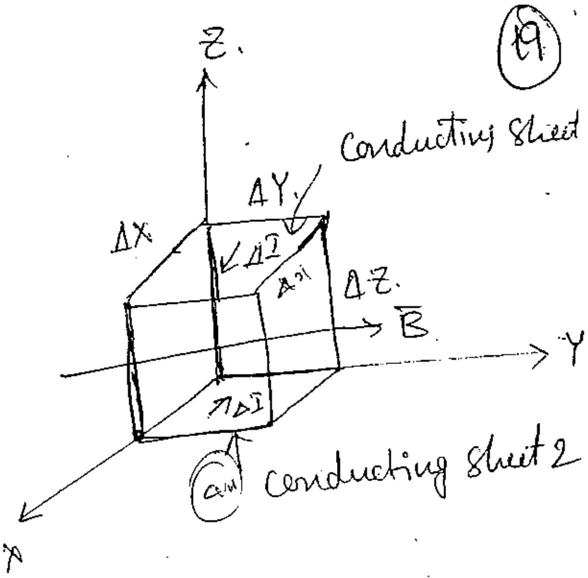
$$= \frac{\mu_0 I}{8\pi} + \int_r^{\infty} \frac{\mu_0 I}{2\pi x} dx$$

$$= \frac{\mu_0 I}{2\pi}\left[\frac{1}{4} + \int_r^{\infty} \frac{dx}{x}\right] \text{ wb-turn/m length}$$

## ENERGY STORED IN A MAGNETIC FIELD:-

inductor energy is stored in magnetic field.

$$\therefore W_m = \frac{1}{2} L I^2$$




consider a differential volume in a magnetic field $\bar{B}$ as shown in fig.

Consider that at the top & bottom surfaces of a differential volume, conducting sheets with $\Delta I$ are present

$$\Delta L = \frac{\Delta \phi}{\Delta I} = \frac{B \Delta S}{\Delta I} \quad \left[\because \phi = \int_S \bar{B} \cdot d\bar{S}\right.$$

$\Delta S = \Delta x \Delta z$ = differential surface area.

$$\Delta L = \frac{B(\Delta x \Delta z)}{\Delta I}$$

$$B = \mu H$$

$$\Delta L = \frac{\mu H \Delta x \Delta z}{\Delta I}$$ — the current flowing through the conducting sheets present at the top & bottom, is in Y direction

$$\Delta I = H(\Delta Y)$$ — energy stored in the inductance differential volume

$$\Delta W_m = \frac{1}{2} \Delta L \, \Delta I^2$$

$$\Delta W_m = \frac{1}{2}\left[\frac{\mu H \Delta x \Delta z}{H \Delta Y}\right][H \Delta Y]^2$$

$$\Delta W_m = \frac{1}{2} \mu H^2 (\Delta x \Delta Y \Delta z)$$

$$\boxed{\Delta W_m = \frac{1}{2} \mu H^2 \Delta V}$$

The magnetostatic energy density function is defined as

$$w_m = \lim_{\Delta V \to 0} \frac{\Delta W_m}{\Delta V} = \frac{1}{2} \mu H^2$$

different forms

$$w_m = \frac{1}{2}(\mu H) H = \frac{1}{2} B H$$

$$= \frac{1}{2} B\left(\frac{B}{\mu}\right) = \frac{B^2}{2\mu}$$

In a linear medium,

$$W_m = \int w_m \, dV$$

$$W_m = \frac{1}{2} \int \bar{B} \cdot \bar{H} \, dV$$

$$= \frac{1}{2} \int \mu H^2 \, dV$$

$$= \frac{1}{2} \int \frac{B^2}{\mu} \, dV$$

# UNIT - V

## TIMEVARYING FIELDS

TIME VARYING FIELDS: UNIT-V



## FARADAY'S LAWS OF ELECTROMAGNETIC INDUCTION:-

$$e = -N\frac{d\phi}{dt}$$

$N=1$ $$e = -\frac{d\phi}{dt}$$

$EMF = \frac{Q_1 Q_2}{4\pi\epsilon_0 d^2}$

$\frac{F}{Q_1} = E$

$$e = \oint \bar{E}\cdot d\bar{L} \quad \text{and} \quad \phi = \int_s \bar{B}\cdot d\bar{s}$$

$$\Rightarrow \quad e = -\frac{d}{dt}\int_s \bar{B}\cdot d\bar{s}$$

$$e = \oint \bar{E}\cdot d\bar{L} = -\frac{d}{dt}\int_s \bar{B}\cdot d\bar{s}$$

* The variation of the flux $\phi$ w.r.t time $t$ can be caused due to any one of the following conditions

(i) By having a stationary closed path in a time varying $\bar{B}$ field ← transformer emf (or) statically induced emf

(ii) " " " time varying closed path in a static $\bar{B}$ field

(iii) " " " " " in a time varying $\bar{B}$ field

→ dynamically induced emf (or) Motional emf.



### A STATIONARY CLOSED PATH IN A TIME VARYING $\bar{B}$ FIELD:

$$\oint \bar{E}\cdot d\bar{L} = -\int_s \frac{\partial \bar{B}}{\partial t}\cdot d\bar{s}$$

($\because$ as $\bar{B}$ may be changing with the co-ordinates as well as time partial derivative is used)

Using Stoke's theorem, line integral can be converted to surface integral

$$\oint_s (\nabla\times\bar{E})\cdot d\bar{s} = -\int_s \frac{\partial \bar{B}}{\partial t}\cdot d\bar{s}$$

assuming that both the surface integrals taken over identical surfaces

$$(\nabla\times\bar{E})\cdot d\bar{s} = -\frac{\partial \bar{B}}{\partial t}\cdot d\bar{s}$$

hence $\boxed{\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}}$

If $\bar{B}$ is not varying with time, then

$$\nabla \times \bar{E} = 0$$

$$\oint E . \bar{dL} = 0$$

i) A MOVING CLOSED PATH IN STATIC $\bar{B}$ FIELD:-

Consider that a charge Q is moved in a magnetic field $\bar{B}$ at a velocity $\bar{V}$. Then the force on a charge is given by

$$\bar{F} = Q\bar{V} \times \bar{B}$$

but the motional EFI is defined as the force/unit charge.

$$\bar{E}_m = \frac{\bar{F}}{Q} = \bar{V} \times \bar{B}$$

thus the induced emf is given by

$$\boxed{\oint \bar{E}_m . \bar{dL} = \oint (\bar{V} \times \bar{B}) . \bar{dL}}$$

The above equation represents total emf induced when a conductor is moved in a uniform const. magnetic field.

ii) MOVING CLOSED PATH IN A TIME VARYING $\bar{B}$ FIELD:

⇒ It is a case in which both the emfs i.e. trf emf & motional emf are present.

⇒ Thus the induced emfs is the combination of these 2 emfs.

Total induced emf = Trf EMF + Motional EMF

i.e., $\boxed{\oint \bar{E} . \bar{dL} = -\int \frac{\partial \bar{B}}{\partial t} . \bar{ds} + \oint (\bar{V} \times \bar{B}) . \bar{dL}}$

## MODIFIED AMPERE'S CIRCUITAL LAW FOR TIME VARYING FIELDS: 2

$$\nabla \times \bar{H} = \bar{J} \rightarrow (1)$$

taking divergence on both sides

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J}$$

but divergence of the curl of any vecter field is zero

thus $\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} = 0 \rightarrow (2)$

but according to current continuity equation

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t} \rightarrow (3)$$

from eqn-(3) $-\frac{\partial \rho_v}{\partial t} = 0$, then eqn (2) becomes true.

thus eqn (2) & (3) are not compatible for time varying fields adding one unknown term

$$\therefore \nabla \times \bar{H} = \bar{J} + \bar{N}$$

taking divergence on both the sides

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} + \nabla \cdot \bar{N}$$

as $\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$, to get correct conditions

$$\nabla \cdot \bar{N} = \frac{\partial \rho_v}{\partial t}$$

but according to Gauss's law $\rho_v = \nabla \cdot \bar{D}$

$$\therefore \nabla \cdot \bar{N} = \frac{\partial}{\partial t}(\nabla \cdot \bar{D})$$

$$\nabla \cdot \bar{N} = \nabla \cdot \frac{\partial \bar{D}}{\partial t}$$

$$\therefore \bar{N} = \frac{\partial \bar{D}}{\partial t}$$

$$\boxed{\therefore \nabla \times \bar{H} = \bar{J}_c + \frac{\partial \bar{D}}{\partial t}}$$

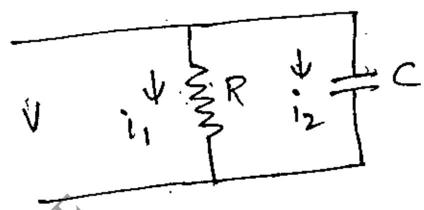
$$\nabla \times \bar{H} = \bar{J}_c + \bar{J}_D$$

where $\bar{J}_c$ = Conduction current density

$\bar{J}_D$ = Displacement current density (as this quantity is obtained from time varying electric flux density)

DISPLACEMENT CURRENT AND DISPLACEMENT CURRENT DENSITY ($\bar{J}_D$)

$$i_1 = \frac{V}{R}$$




this is called conduction current
(because of actual motion of charges)

Let 'A' is the cross-sectional area of R,

then $\bar{J}_c = \frac{i_c}{A} = \sigma \bar{E} \rightarrow (1)$

⇒ Now assume that the initial charge on a capacitor is zero. then for time varying voltage applied across parallel plate capacitor, then

$$i_2 = C \frac{dV}{dt}$$

Let A - area
d - distance
ε = dielectric

$$i_2 = \frac{\varepsilon A}{d} \frac{dV}{dt}$$

this current is called displacement current

$$E = \frac{V}{d} \qquad V = -\int \bar{E} \cdot \bar{dl}$$

$$V = dE$$

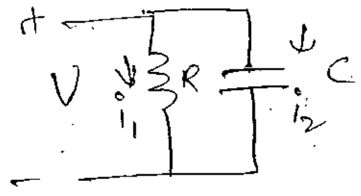
$$\therefore i_2 = \frac{\varepsilon A}{d} \frac{d}{dt}(dE)$$

$$i_D = \frac{\varepsilon A}{d} d \frac{dE}{dt}$$ – as distance d is not varying with time

$$i_D = \varepsilon A \frac{dE}{dt} \Rightarrow \bar{J}_D = \frac{i_D}{A} = \frac{\varepsilon A \frac{dE}{dt}}{A} = \frac{d\,\varepsilon E}{dt} = \frac{d\bar{D}}{dt}$$

# DISPLACEMENT CURRENT & DISPLACEMENT CURRENT DENSITY ($\bar{J}_D$) ③

$$\bar{J}_C = \frac{i_c}{A} = \sigma\bar{E}$$




$$i_2 = C\frac{dV}{dt}$$

$$i_2 = \frac{\varepsilon A}{d}\frac{dV}{dt}$$

$$E = \frac{V}{d}$$

$$V = (d)(E)$$

$$i_D = i_2 = \frac{\varepsilon A}{d}\cdot\frac{d}{dt}(dE)$$

$$i_D = \frac{\varepsilon A}{d}\cdot d\frac{dE}{dt}$$

$$= \varepsilon A\frac{dE}{dt}$$

$$\bar{J}_D = \frac{i_D}{A} = \frac{\varepsilon \cancel{A}}{\cancel{A}}\frac{d\bar{E}}{dt}$$

$$= \frac{d}{dt}(\varepsilon\bar{E})$$

$$\bar{J}_D = \frac{\partial\bar{D}}{\partial t}$$

For $\bar{E}$, let the time dependence be given by $e^{j\omega t}$, the total current density

$$\bar{J} = \bar{J}_c + \bar{J}_D = \sigma\bar{E} + \frac{\partial\bar{D}}{\partial t}$$

$$= \sigma\bar{E} + j\omega\varepsilon\bar{E}$$

$$\frac{|\bar{J}_c|}{|\bar{J}_D|} = \frac{\sigma}{\omega\varepsilon}$$

$\frac{\sigma}{\omega\varepsilon} >> 1$ for conductors

$\frac{\sigma}{\omega\varepsilon} << 1$ for dielectric

Thus in a given medium both the currents $i_c$ & $i_d$ may flow

$$\bar{J}_C = \bar{E}\sigma$$

$$\bar{J}_D = \frac{\partial\bar{D}}{\partial t}$$

total $\bar{J} = \bar{J}_c + \bar{J}_D$

Maxwell's equations for time varying fields

$$\oint \bar{H} \cdot d\bar{L} = I_{encl}$$

$$\oint \bar{H} \cdot d\bar{L} = \int_S \bar{J} \cdot d\bar{S} \quad (3)$$

$$\oint \bar{H} \cdot d\bar{L} = \int_S \left[\bar{J} + \frac{\partial D}{\partial t}\right] \cdot d\bar{S} \rightarrow (1)$$

→ The net mmf around a closed path is equal to the surface integral of the conduction and displacement current densities over the entire surface bounded by the closed path.

applying stoke's theorem to LHS of eqn (1)

$$\int_S (\nabla \times \bar{H}) \cdot d\bar{S} = \int_S \left[\bar{J} + \frac{\partial \bar{D}}{\partial t}\right] \cdot d\bar{S}$$

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \rightarrow (2) \quad \text{(point form)}$$

---

according to faraday's law

(√) $$\oint E \cdot d\bar{L} = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{S} \rightarrow (3)$$

→ The net emf around a closed path is equal to the surface integral of the negative time rate of change of the magnetic flux density over the surface bounded by the closed path.

using stoke's theorem

$$\int_S (\nabla \times \bar{E}) \cdot d\bar{S} = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{S}$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad \text{— (point form)}$$

ccording to gauss law

$$\int_S \bar{D} \cdot \bar{ds} = Q_{encl}$$

1

$$\int_S \bar{D} \cdot \bar{ds} = \int_V \rho_V \, dV \rightarrow (5)$$

The net electric flux emerging out of a closed surface enclosing the volume is equal to the net charge enclosed.

using divergence theorem

$$\int_V (\nabla \cdot \bar{D}) \cdot dV = \int_V \rho_V \, dV$$

$$\nabla \cdot \bar{D} = \rho_V \rightarrow (6) \text{ point form}$$

for magnetic fields,

$$\int_S \bar{B} \cdot \bar{dS} = 0 \rightarrow (7)$$

(J)

⇒ The net magnetic flux emerging out the closed surface is equal to zero.

using divergence theorem

$$\int (\nabla \cdot \bar{B}) \, dV = 0 \rightarrow (8)$$

$$\nabla \cdot \bar{B} = 0 \rightarrow (8) \text{ point form.}$$

| Integral form | differential form | Significance |
|---|---|---|
| $\oint \bar{E}\cdot d\bar{l} = -\int_S \frac{\partial \bar{B}}{\partial t}\cdot d\bar{s}$ | $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$ | Faraday's law |
| $\oint \bar{H}\cdot d\bar{l} = \cancel{I}\int_S \frac{\partial \bar{D}}{\partial t}\cdot d\bar{s}$ | $\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$ | Ampere's circuital law |
| $\oint_S \bar{D}\cdot d\bar{s} = \int_V \rho_V\, dV$ | $\nabla\cdot\bar{D} = \rho_V$ | Gauss law |
| $\oint_S \bar{B}\cdot d\bar{s} = 0$ | $\nabla\cdot\bar{B} = 0$ | No isolated magnetic charge. |

For free space: $\rho_V = 0$, $\sigma = 0$

| Point form | Integral form |
|---|---|
| $\nabla\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}$ | $\oint \bar{E}\cdot d\bar{l} = -\int_S \frac{\partial \bar{B}}{\partial t}\cdot d\bar{s}$ |
| $\nabla\times\bar{H} = \frac{\partial \bar{D}}{\partial t}$ | $\oint \bar{H}\cdot d\bar{l} = \int_S \frac{\partial \bar{D}}{\partial t}\cdot d\bar{s}$ |
| $\nabla\cdot\bar{D} = 0$ | $\oint_S \bar{D}\cdot d\bar{s} = 0$ |
| $\nabla\cdot\bar{B} = 0$ | $\oint_S \bar{B}\cdot d\bar{s} = 0$ |

For Good Conductors: $\bar{J} >> \frac{\partial \bar{D}}{\partial t}$; $\rho_V = 0$

| | |
|---|---|
| $\nabla\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}$ | $\oint \bar{E}\cdot d\bar{l} = -\int_S \frac{\partial \bar{B}}{\partial t}\cdot d\bar{s}$ |
| $\nabla\times\bar{H} = \bar{J}$ | $\oint \bar{H}\cdot d\bar{l} = \cancel{I}\int_S \bar{J}\cdot d\bar{s}$ |
| $\nabla\cdot\bar{D} = 0$ | $\oint \bar{D}\cdot d\bar{s} = 0$ |
| $\nabla\cdot\bar{B} = 0$ | $\oint_S \bar{B}\cdot d\bar{s} = 0$ |

For Harmonically Varying fields:

$\bar{D} = \bar{D}_0 e^{j\omega t}$ ; $\bar{B} = \bar{B}_0 e^{j\omega t}$

taking partial derivative w.r.t time.

$\frac{\partial \bar{D}}{\partial t} = j\omega \bar{D}_0 e^{j\omega t} = j\omega\bar{D}$

$\frac{\partial \bar{B}}{\partial t} = j\omega \bar{B}_0 e^{j\omega t} = j\omega\bar{B}$

| POINT form | Integral form |
|---|---|
| $\nabla\times\bar{E} = -j\omega\bar{B} = -j\omega\mu\bar{H}$ | $\oint \bar{E}\cdot d\bar{l} = -\int_S j\omega\bar{B}\cdot d\bar{s}$ $= -\int_S j\omega\mu\bar{H}\, d\bar{s}$ $= -j\omega\mu\int_S \bar{H}\cdot d\bar{s}$ |
| $\nabla\times\bar{H} = \bar{J} + j\omega\bar{D}$ $= \sigma\bar{E} + j\omega(\varepsilon\bar{E})$ $= (\sigma + j\omega\varepsilon)\bar{E}$ | $\oint \bar{H}\cdot d\bar{l} = I + \int_S j\omega\bar{D}\cdot d\bar{s}$ $= \int_S \bar{J}\cdot d\bar{s} + \int_S j\omega\varepsilon\bar{E}\cdot d\bar{s}$ $= \int_S \sigma\bar{E}\cdot d\bar{s} + \int_S j\omega\varepsilon\bar{E}\cdot d\bar{s}$ $= (\sigma + j\omega\varepsilon)\int_S \bar{E}\cdot d\bar{s}$ |
| $\nabla\cdot\bar{D} = \rho_V$ | $\oint_S \bar{D}\cdot d\bar{s} = \oint_V \rho_V\, dV$ |
| $\nabla\cdot\bar{B} = 0$ | $\oint_S \bar{B}\cdot d\bar{s} = 0$ |

## POYNTING THEOREM:

From Max well's second equation for time-varying fields

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \bar{E} = -\mu \frac{\partial H}{\partial t} \rightarrow (1)$$

from III eqn.

$$\nabla \times \bar{H} = \sigma \bar{E} + \frac{\partial \bar{D}}{\partial t}$$

$$\nabla \times \bar{H} = \sigma \bar{E} + \varepsilon \frac{\partial \bar{E}}{\partial t}$$

Dotting both sides of above eqn. with $\bar{E}$

$$\bar{E}\cdot(\nabla \times \bar{H}) = \sigma \bar{E}^2 + \bar{E}\cdot \varepsilon \frac{\partial \bar{E}}{\partial t}$$

but for any vector fields A & B

$$\nabla\cdot(A \times B) = B\cdot(\nabla \times A) - A\cdot(\nabla \times B)$$

$A = \bar{H}$ & $B = \bar{E}$

$$\nabla\cdot(\bar{H}\times\bar{E}) = \bar{E}\cdot(\nabla\times\bar{H}) - \bar{H}\cdot(\nabla\times\bar{E})$$

$$\bar{E}\cdot(\nabla\times\bar{H}) = \nabla\cdot(\bar{H}\times\bar{E}) + \bar{H}\cdot(\nabla\times\bar{E})$$

$$\frac{\partial}{\partial t}(H)^2 = 2H$$

$$\sigma\bar{E}^2 + \bar{E}\cdot\varepsilon\frac{\partial\bar{E}}{\partial t} = \nabla\cdot(\bar{H}\times\bar{E}) + \bar{H}\cdot(\nabla\times\bar{E})$$

$$\bar{H}\cdot\frac{\partial \bar{H}}{\partial t}$$

$$\bar{H}\cdot(\nabla\times\bar{E}) = \bar{H}\cdot\left(-\mu\frac{\partial\bar{H}}{\partial t}\right) = -\frac{\mu}{2}\frac{\partial}{\partial t}(\bar{H}\cdot\bar{H})$$

$$-\frac{\mu}{2}\frac{\partial H^2}{\partial t} - \nabla\cdot(\bar{E}\times\bar{H}) = \sigma\bar{E}^2 + \frac{1}{2}\varepsilon\frac{\partial\bar{E}^2}{\partial t}$$

Rearranging and taking volume integrals.

$$\int_V \nabla\cdot(\bar{E}\times\bar{H})\,dV = -\frac{\partial}{\partial t}\iint_V\left(\frac{1}{2}\varepsilon\bar{E}^2 + \frac{1}{2}\mu\bar{H}^2\right)dV - \int_V \sigma\bar{E}^2 dV$$

applying the divergence theorem to LHS

$$\oint_S (\bar{E} \times \bar{H}) \cdot d\bar{s} = -\frac{\partial}{\partial t} \int_V \left[\frac{1}{2} \varepsilon \bar{E}^2 + \frac{1}{2} \mu \bar{H}^2\right] dv - \int_V \sigma \bar{E}^2 dV$$

Total power leaving the volume = Rate of decrease in energy stored in electric & magnetic fields − ohmic power dissipated.

The above equation is referred to as ~~point~~ poynting's theorem

The quantity $\bar{E} \times \bar{H}$ is known as the poynting vector in watt/sq.m

$$P = \bar{E} \times \bar{H} \quad w/m^2$$

It represents the instantaneous power density vector associated with the EM field at a given point. The integration of the poynting vector over any closed surface gives the net power flowing out of that surface.

poynting's theorem states that the net power flowing out of a given volume 'v' is equal to the time rate of decrease in the energy stored with in 'v' minus the conduction losses.

UNIT - V - problems

(1) In a material for which $\sigma = 5.0$ S/m and $\varepsilon_r = 1$, the electric field intensity is $E = 250 \sin 10^{10} t$ V/m. Find the conduction & displacement current densities and the freq at which both have equal magnitude.

Ans)
$$J_C = \sigma E$$
$$= 5 (250 \sin 10^{10} t)$$
$$= 1250 \sin 10^{10} t \quad A/m^2$$
$$J_D = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\varepsilon E)$$
$$= \frac{\partial}{\partial t}(\varepsilon_0 \varepsilon_r E)$$
$$= \frac{\partial}{\partial t}(8.854 \times 10^{-12} \times 1 \times 250 \sin 10^{10} t)$$
$$= 22.135 \cos 10^{10} t \quad A/m^2$$
$$\frac{J_c}{J_D} = \frac{\sigma}{\varepsilon \omega} = 1$$
$$\omega = \frac{\sigma}{\varepsilon} = \frac{5}{8.854 \times 10^{-12} \times 1} = 5.6471 \times 10^{11}$$
$$\omega = 2\pi f.$$
$$f = \frac{\omega}{2\pi} = \frac{5.6471 \times 10^{11}}{2\pi} = 89.87 \text{ GHz}$$

VARYING FIELDS: UNIT-V (1)

## FARADAY'S LAWS OF ELECTROMAGNETIC INDUCTION:-

$$e = -N\frac{d\phi}{dt}$$

$N=1$ $\quad e = -\frac{d\phi}{dt}$

$EMF = \frac{Q_1 Q_2}{4\pi\epsilon_0 d^2}$ $\quad F/Q = E$

$$e = \oint \bar{E}\cdot d\bar{L} \quad \text{and} \quad \phi = \int_s \bar{B}\cdot d\bar{S}$$

$$\Rightarrow \quad e = -\frac{d}{dt}\int_s \bar{B}\cdot d\bar{S}$$

$$e = \oint \bar{E}\cdot d\bar{L} = -\frac{d}{dt}\int_s \bar{B}\cdot d\bar{S}$$

* The variation of the flux $\phi$ w.r.t time $t$ can be caused due to any one of the following conditions

(i) By having a stationary closed path in a time varying $\bar{B}$ field ← transformer emf (or) statically induced emf

(ii) " " " time varying closed path in a static $\bar{B}$ field

(iii) " " " " " in a time varying $\bar{B}$ field

↓ dynamically induced emf & Motional emf.

## A STATIONARY CLOSED PATH IN A TIME VARYING $\bar{B}$ FIELD:

$$\oint \bar{E}\cdot d\bar{L} = -\int_s \frac{\partial \bar{B}}{\partial t}\cdot d\bar{S}$$

( ∵ as $\bar{B}$ may be changing with the co-ordinates as well as time partial derivative is used)

Using Stokes theorem, line integral can be converted to surface integral

$$\oint_s (\nabla \times \bar{E})\cdot d\bar{S} = -\int_s \frac{\partial \bar{B}}{\partial t}\cdot d\bar{S}$$

assuming that both the surface integrals taken over identical surface

$$(\nabla \times \bar{E})\cdot d\bar{S} = -\frac{\partial \bar{B}}{\partial t}\cdot d\bar{S}$$

hence $\boxed{\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}}$

If $\bar{B}$ is not varying with time, then

$$\nabla \times \bar{E} = 0$$

$$\oint E \cdot d\bar{L} = 0$$

i) A MOVING CLOSED PATH IN STATIC $\bar{B}$ FIELD:-

Consider that a charge Q is moved in a magnetic field $\bar{B}$ at a velocity $\bar{V}$. Then the force on a charge is given by

$$\bar{F} = Q\bar{V} \times \bar{B}$$

but the motional EFI is defined as the force/unit charge.

$$\bar{E}_m = \frac{\bar{F}}{Q} = \bar{V} \times \bar{B}$$

Thus the induced emf is given by

$$\boxed{\oint \bar{E}_m \cdot d\bar{L} = \oint (\bar{V} \times \bar{B}) \cdot d\bar{L}}$$

The above equation represents total emf induced when a conductor is moved in a uniform const. magnetic field.

ii) MOVING CLOSED PATH IN A TIME VARYING $\bar{B}$ FIELD:

⇒ It is a case in which both the emfs i.e. trf emf & motional emf are present.

⇒ Thus the induced emfs is the combination of these 2 emfs.

Total induced emf = Trf EMF + Motional EMF

i.e., $\boxed{\oint \bar{E} \cdot d\bar{L} = -\int \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s} + \oint (\bar{V} \times \bar{B}) \cdot d\bar{L}}$

## MODIFIED AMPERE'S CIRCUITAL LAW FOR TIME VARYING FIELDS: (2)

$$\nabla \times \bar{H} = \bar{J} \rightarrow (1)$$

taking divergence on both sides

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J}$$

but divergence of the curl of any vector field is zero

thus $\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} = 0 \rightarrow (2)$

but according to current continuity equation

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t} \rightarrow (3)$$

from eqn-(3) $-\frac{\partial \rho_v}{\partial t} = 0$, then eqn (2) becomes true.

thus eqn (2) & (3) are not compatible for time varying fields
adding one unknown term

$$\therefore \nabla \times \bar{H} = \bar{J} + \bar{N}$$

taking divergence on both the sides

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} + \nabla \cdot \bar{N}$$

as $\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$, to get correct conditions

$$\nabla \cdot \bar{N} = \frac{\partial \rho_v}{\partial t}$$

but according to Gauss's law $\rho_v = \nabla \cdot \bar{D}$

$$\therefore \nabla \cdot \bar{N} = \frac{\partial}{\partial t}(\nabla \cdot \bar{D})$$

$$\nabla \cdot \bar{N} = \nabla \cdot \frac{\partial \bar{D}}{\partial t}$$

$$\therefore \bar{N} = \frac{\partial \bar{D}}{\partial t}$$

$$\boxed{\nabla \times \bar{H} = \bar{J}_c + \frac{\partial \bar{D}}{\partial t}}$$

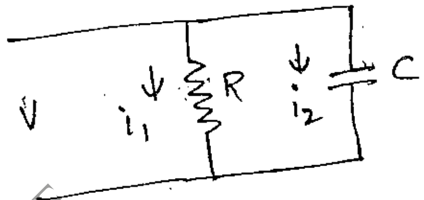
$$\nabla \times \bar{H} = \bar{J}_c + \bar{J}_D$$

where $\bar{J}_c$ = Conduction current density

$\bar{J}_D$ = Displacement current density (as this quantity is obtained from time varying electric flux density)

DISPLACEMENT CURRENT AND DISPLACEMENT CURRENT DENSITY ($\bar{J}_D$)

$$i_1 = \frac{V}{R}$$




this is called Conduction current (because of actual motion of charges)

Let 'A' is the cross-sectional area of R,

then $\bar{J}_c = \frac{i_c}{A} = \sigma \bar{E} \rightarrow (1)$

⇒ Now assume that the initial charge on a capacitor is zero. then for time varying voltage applied across parallel plate capacitor, then

$$i_2 = C \frac{dV}{dt}$$

Let A - area
d - distance
$\varepsilon$ = dielectric

$$i_2 = \frac{\varepsilon A}{d} \frac{dV}{dt}$$

this current is called displacement current

$$E = \frac{V}{d}$$

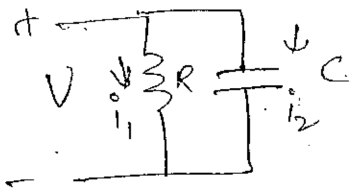
$$V = dE$$

$$\therefore \; i_2 = \frac{\varepsilon A}{d} \frac{d}{dt}(dE)$$

$$i_D = \frac{\varepsilon A}{d} d \frac{dE}{dt}$$ — as distance d is not varying with time

$$i_D = \varepsilon A \frac{dE}{dt} \Rightarrow \bar{J}_D = \frac{i_D}{A} = \frac{\varepsilon A \frac{dE}{dt}}{A} = \frac{d\,\varepsilon E}{dt} = \frac{d\bar{D}}{dt}$$

## DISPLACEMENT CURRENT & DISPLACEMENT CURRENT DENSITY ($\bar{J}_D$) (3)




$$\bar{J}_c = \frac{i_c}{A} = \sigma \bar{E}$$

$$i_2 = C\frac{dv}{dt}$$

$$i_2 = \frac{\varepsilon A}{d}\frac{dv}{dt}$$

$$E = \frac{V}{d}$$

$$V = (d)(E)$$

$$i_D = i_2 = \frac{\varepsilon A}{d}\cdot\frac{d}{dt}(dE)$$

$$i_D = \frac{\varepsilon A}{d}\cdot d\,\frac{dE}{dt}$$

$$= \varepsilon A\frac{dE}{dt}$$

$$\bar{J}_D = \frac{i_D}{A} = \frac{\varepsilon \cancel{A}}{\cancel{A}}\frac{dE}{dt}$$

$$= \frac{d}{dt}(\varepsilon \bar{E})$$

$$\bar{J}_D = \frac{\partial \bar{D}}{\partial t}$$

For $\bar{E}$, let the time dependence be given by $e^{j\omega t}$, the total current density

$$\bar{J} = \bar{J}_c + \bar{J}_D = \sigma\bar{E} + \frac{\partial \bar{D}}{\partial t}$$

$$= \sigma\bar{E} + j\omega\varepsilon\bar{E}$$

$$\frac{|\bar{J}_c|}{|\bar{J}_D|} = \frac{\sigma}{\omega\varepsilon}$$

$\frac{\sigma}{\omega\varepsilon} >> 1$ for conductors

$\frac{\sigma}{\omega\varepsilon} << 1$ for dielectric

Thus in a given medium both the currents $i_c$ & $i_d$ may flow

$$\bar{J}_c = \bar{E}\sigma$$

$$\bar{J}_D = \frac{\partial \bar{D}}{\partial t}$$

total $\bar{J} = \bar{J}_c + \bar{J}_D$

Maxwell's equations for time varying fields

$$\oint \bar{H}\cdot d\bar{L} = I_{encl}$$

$$\oint \bar{H}\cdot d\bar{L} = \int_S \bar{J}\cdot d\bar{S} \quad ?$$

$$\oint \bar{H}\cdot d\bar{L} = \int_S \left[\bar{J} + \frac{\partial D}{\partial t}\right]\cdot d\bar{S} \rightarrow (1)$$

→ The net ~~emf~~ mmf around a closed path is equal to the surface integral of the conduction and displacement current densities over the entire surface bounded by the closed path.

applying stoke's theorem to LHS of eqn (1)

$$\int_S (\nabla\times\bar{H})\cdot d\bar{S} = \int_S \left[\bar{J} + \frac{\partial\bar{D}}{\partial t}\right]\cdot d\bar{S}$$

$$\nabla\times\bar{H} = \bar{J} + \frac{\partial\bar{D}}{\partial t} \rightarrow (2) \quad \text{(point form)}$$

---

according to faraday's law:

$$\text{(IV)} \quad \oint \bar{E}\cdot d\bar{L} = -\int_S \frac{\partial\bar{B}}{\partial t}\cdot d\bar{S} \rightarrow (3)$$

→ The net emf around a closed path is equal to the surface integral of the negative time rate of change of the magnetic flux density over the surface bounded by the closed path.

using stoke's theorem

$$\int_S (\nabla\times\bar{E})\cdot d\bar{S} = -\int_S \frac{\partial\bar{B}}{\partial t}\cdot d\bar{S}$$

$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} \quad \text{— (point form)}$$

cording to gauss law

$$\int_S \bar{D} \cdot \bar{ds} = Q_{encl}$$

(1)

$$\int_S \bar{D} \cdot \bar{ds} = \int_V \rho_V \, dV \rightarrow (5)$$

The net electric flux emerging out of a closed surface enclosing the volume is equal to the net charge enclosed.

using divergence theorem

$$\int_V (\nabla \cdot \bar{D}) \cdot dV = \int_V \rho_V \, dV$$

$$\nabla \cdot \bar{D} = \rho_V \rightarrow (6) \text{ point form}$$

for magnetic fields,

$$\int_S \bar{B} \cdot \bar{ds} = 0 \rightarrow (7)$$

(2)

The net magnetic flux emerging out the closed surface is equal to zero.

using divergence theorem

$$\int (\nabla \cdot \bar{B}) \, dV = 0 \rightarrow (\cancel{8})$$

$$\nabla \cdot \bar{B} = 0 \rightarrow (8) \text{ point form.}$$

| Integral form | differential form | Significance |
|---|---|---|
| $\oint \bar{E}\cdot d\bar{l} = -\int_s \frac{\partial \bar{B}}{\partial t}\cdot d\bar{s}$ | $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$ | Faraday's law |
| $\oint \bar{H}\cdot d\bar{l} = I + \int_s \frac{\partial \bar{D}}{\partial t}\cdot d\bar{s}$ | $\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$ | Ampere's circuital law |
| $\oint_s \bar{D}\cdot d\bar{s} = \int_v \rho_v \, dv$ | $\nabla \cdot \bar{D} = \rho_v$ | Gauss law |
| $\oint_s \bar{B}\cdot d\bar{s} = 0$ | $\nabla \cdot \bar{B} = 0$ | No isolated magnetic charge. |

For free space: $\rho_v = 0$

$\sigma = 0$

| Point form | Integral form |
|---|---|
| $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$ | $\oint \bar{E}\cdot d\bar{l} = -\int_s \frac{\partial \bar{B}}{\partial t}\cdot d\bar{s}$ |
| $\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t}$ | $\oint \bar{H}\cdot d\bar{l} = \int_s \frac{\partial \bar{D}}{\partial t}\cdot d\bar{s}$ |
| $\nabla \cdot \bar{D} = 0$ | $\oint_s \bar{D}\cdot d\bar{s} = 0$ |
| $\nabla \cdot \bar{B} = 0$ | $\oint_s \bar{B}\cdot d\bar{s} = 0$ |

For Good Conductors: $\bar{J} >> \frac{\partial \bar{D}}{\partial t}$; $\rho_v = 0$

| | |
|---|---|
| $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$ | $\oint \bar{E}\cdot d\bar{l} = -\int_s \frac{\partial \bar{B}}{\partial t}\cdot d\bar{s}$ |
| $\nabla \times \bar{H} = \bar{J}$ | $\oint \bar{H}\cdot d\bar{l} = I + \int_s \bar{J}\cdot d\bar{s}$ |
| $\nabla \cdot \bar{D} = 0$ | $\oint \bar{D}\cdot d\bar{s} = 0$ |
| $\nabla \cdot \bar{B} = 0$ | $\oint_s \bar{B}\cdot d\bar{s} = 0$ |

For Harmonically Varying fields:

$\bar{D} = \bar{D}_0 e^{j\omega t}$ ; $\bar{B} = \bar{B}_0 e^{j\omega t}$

taking partial derivative wrt time.

$\frac{\partial \bar{D}}{\partial t} = j\omega \bar{D}_0 e^{j\omega t} = j\omega \bar{D}$

$\frac{\partial \bar{B}}{\partial t} = j\omega \bar{B}_0 e^{j\omega t} = j\omega \bar{B}$

| POINT form | Integral form |
|---|---|
| $\nabla \times \bar{E} = -j\omega \bar{B} = -j\omega\mu \bar{H}$ <br> $\nabla \times \bar{H} = \bar{J} + j\omega \bar{D}$ <br> $= \sigma \bar{E} + j\omega(\varepsilon \bar{E})$ <br> $= (\sigma + j\omega\varepsilon)\bar{E}$ <br> $\nabla \cdot \bar{D} = \rho_v$ <br> $\nabla \cdot \bar{B} = 0$ | $\oint \bar{E}\cdot d\bar{l} = -\int_s j\omega \bar{B}\cdot d\bar{s}$ <br> $= -\int_s j\omega\mu \bar{H}\, d\bar{s}$ <br> $= -j\omega\mu \int_s \bar{H}\cdot d\bar{s}$ <br> $\oint \bar{H}\cdot d\bar{l} = I + \int_s j\omega \bar{D}\, d$ <br> $= \int_s \bar{J}\cdot d\bar{s} + \int_s j\omega\varepsilon \bar{E}\cdot d$ <br> $= \int_s \sigma \bar{E}\cdot d\bar{s} + \int_s j\omega\varepsilon \bar{E}\cdot d$ <br> $= (\sigma + j\omega\varepsilon)\int_s \bar{E}\cdot d\bar{s}$ |
| $\oint_s \bar{D}\cdot d\bar{s} = \oint_v \rho_v \, dv$ | |
| $\oint_s \bar{B}\cdot d\bar{s} = 0$ | |

## POYNTING THEOREM:

From Max well's second equation for time-varying fields

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \bar{E} = -\mu \frac{\partial H}{\partial t} \rightarrow 1$$

From III eqn.

$$\nabla \times \bar{H} = \sigma \bar{E} + \frac{\partial \bar{D}}{\partial t}$$

$$\nabla \times \bar{H} = \sigma \bar{E} + \varepsilon \frac{\partial \bar{E}}{\partial t}$$

Dotting both sides of above eqn. with $\bar{E}$

$$\bar{E} \cdot (\nabla \times \bar{H}) = \sigma \bar{E}^2 + \bar{E} \cdot \varepsilon \frac{\partial E}{\partial t}$$

but for any vector fields A & B

$$\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$$

$A = \bar{H}$ & $B = \bar{E}$

$$\nabla \cdot (\bar{H} \times \bar{E}) = \bar{E} \cdot (\nabla \times \bar{H}) - \bar{H} \cdot (\nabla \times \bar{E})$$

$$\bar{E} \cdot (\nabla \times \bar{H}) = \nabla \cdot (\bar{H} \times \bar{E}) + \bar{H} \cdot (\nabla \times \bar{E})$$

$$\frac{\partial}{\partial t}(H)^2 = 2H$$

$$\sigma \bar{E}^2 + \bar{E} \cdot \varepsilon \frac{\partial \bar{E}}{\partial t} = \nabla \cdot (\bar{H} \times \bar{E}) + \bar{H} \cdot (\nabla \times \bar{E})$$

$$\bar{H} \cdot (\nabla \times \bar{E}) = \bar{H} \cdot \left(-\mu \frac{\partial \bar{H}}{\partial t}\right) = -\frac{\mu}{2} \frac{\partial}{\partial t}(\bar{H} \cdot \bar{H})$$

$$H \cdot \frac{\partial H}{\partial t}$$

$$-\frac{\mu}{2} \frac{\partial H^2}{\partial t} - \nabla \cdot (\bar{E} \times \bar{H}) = \sigma \bar{E}^2 + \frac{1}{2} \varepsilon \frac{\partial \bar{E}^2}{\partial t}$$

Rearranging and taking volume integrals

$$\int_V \nabla \cdot (\bar{E} \times \bar{H})\, dV = -\frac{\partial}{\partial t} \iint_V \left(\frac{1}{2} \varepsilon \bar{E}^2 + \frac{1}{2} \mu \bar{H}^2\right) dV - \int_V \sigma \bar{E}^2 dV$$

applying the divergence theorem to LHS

$$\oint_S (\bar{E} \times \bar{H}) \cdot \bar{ds} = -\frac{\partial}{\partial t} \iint_V \left[\frac{1}{2} \varepsilon \bar{E}^2 + \frac{1}{2} \mu \bar{H}^2\right] dV - \int_V \sigma E^2 dV$$

Total power leaving the volume = Rate of decrease in energy stored in electric & magnetic fields − ohmic power dissipated.

The above equation is referred to as ~~point~~ poynting's theorem

The quantity $\bar{E} \times \bar{H}$ is known as the poynting vector in watt/sq.m

$$P = \bar{E} \times \bar{H} \quad W/m^2$$

It represents the instantaneous power density vector associated with the EM field at a given point. The integration of the poynting vector over any closed surface gives the net power flowing out of that surface.

poynting's theorem states that the net power flowing out of a given volume 'v' is equal to the time rate of decrease in the energy stored within 'v' minus the conduction losses.

(3) In cylindrical coordinates $\bar{B} = \frac{2}{r}\bar{a}_\phi$ T. Determine $\phi$, crossing the plane surface defined by $0.5 \le r \le 2.5$ m and $0 \le z \le 2$ m.

Sol:

$$\phi = \int_s \bar{B}\cdot\bar{ds} = \int_{z=0}^{2}\int_{r=0.5}^{2.5} \frac{2}{r}\,dr\,dz$$

$$= 2.\left[\ln r\right]_{0.5}^{2.5}\left[z\right]_0^2$$

$$= 2\left[\ln 2.5 - \ln 0.5\right](2-0)$$

$$= 6.4377 \text{ wb}$$

(4) Calculate the inductance of a solenoid of 200 turns wound on a cylindrical tube of 6cm diameter. The length of the tube is 60cm & the solenoid is in air.

Sol:

$$L = \frac{\mu N^2 A}{l}$$

$$= \frac{4\pi\times10^{-7}\times(200)^2\times\pi\times(3\times10^{-2})^2}{60\times10^{-2}}$$

$$= 0.2368 \text{ mH}$$

② A lossy dielectric has $\mu_r = 1.5$, $\varepsilon_r = 1$, $\sigma = 2\times10^{-8}$ s/m and $\bar{E} = 100 \sin\omega t \, \bar{a}_z$ V/m

(i) Determine at what frequency the conduction & displacement currents are equal

(ii) Calculate instantaneous displacement current density at that frequency.

(A)

① $\dfrac{\sigma}{\omega\varepsilon} = 1$

$$\omega = \frac{2\times10^{-8}}{8.854\times10^{-12}\times1} = 2258.86 \text{ rad/sec}$$

$$f = \frac{\omega}{2\pi} = \frac{2258.86}{2\pi} = 359.5 \text{ Hz}$$

② $$J_D = \frac{d}{dt}\varepsilon E = \frac{d}{dt}\left(8.854\times10^{-12}\times 1\times 100\sin\omega t\right)$$

$$= 8.854\times10^{-12}\times1\left[\frac{d}{dt}100\sin\omega t\right]$$

$$= 8.854\times10^{-12}\times1\times100\times\omega\times\cos\omega t$$

$$= 8.854\times10^{-12}\times1\times100\times 2\times\pi\times359.5\times\cos(2\pi359.5)t$$

$$= 1.9999\times10^{-6}\cos 719\pi t.$$

$$= 2\cos 719\pi t \ \mu A/m^2$$

1. A solenoid of 500 turns has a length of 50cm and the radius of 10cm. A steel rod of circular cross-section is fitted in the solenoid coaxially. Relative permeability of steel is 3000. A DC current of 10A is passed through solenoid. Compute inductance of the system, energy stored in the system and mean flux density inside the solenoid.

sol:

$N = 500$

$l = 50 \times 10^{-2} = 0.5m$

$r = 10 \times 10^{-2} = 0.1m \qquad = A = \pi r^2$

$\mu_{r_s} = 3000$

$I = 10 A.$

$\cancel{H = \frac{NI}{l}} \Rightarrow$ (1) $L = \dfrac{\mu_0 \mu_r N^2 A}{l}$

$= \dfrac{4\pi \times 10^{-7} \times 3000 \times 500^2 \times (\pi \times (0.1)^2)}{0.5}$

$= 59.2176 \ H$

(2) $W_H = \frac{1}{2} L I^2 = \frac{1}{2} [59.21] [10]^2$

$= 2.96 \ KJ$

(3) $B = \dfrac{\mu_0 \mu_r N I}{l}$ $\qquad \bar{H} = \dfrac{NI}{l}$ & $\bar{B} = \mu \bar{H}$

$= \dfrac{4\pi \times 10^{-7} \times 3000 \times 500 \times 10}{0.5}$

$= 37.69 \ Wb/m^2$

## MAXWELL'S EQUATIONS FOR GOOD CONDUCTOR:

$$\bar{J} >> \frac{\partial \bar{D}}{\partial t} \quad \& \quad \rho_v = 0$$

**POINT FORM**

(1) $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

(2) $\nabla \times \bar{H} = \bar{J}$

(3) $\nabla \cdot \bar{D} = 0$

(4) $\nabla \cdot \bar{B} = 0$

**INTEGRAL FORM**

(1) $\oint \bar{E} \cdot d\bar{l} = -\int_s -\frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}$

(2) $\oint \bar{H} \cdot d\bar{l} = I = \int_s \bar{J} \cdot d\bar{s}$

(3) $\oint_s \bar{D} \cdot d\bar{s} = 0$

(4) $\oint_s \bar{B} \cdot d\bar{s} = 0$

## MAXWELL'S EQUATIONS FOR HARMONICALLY VARYING FIELDS:

assume that electric & magnetic fields are varying harmonically with time.

The electric flux density $\bar{D} = \bar{D}_0 e^{j\omega t}$

magnetic flux density $\bar{B} = \bar{B}_0 e^{j\omega t}$

taking partial derivative w.r.t. time.

$$\frac{\partial \bar{D}}{\partial t} = j\omega \bar{D}_0 e^{j\omega t} = j\omega \bar{D}$$

$$\frac{\partial \bar{B}}{\partial t} = j\omega \bar{B}_0 e^{j\omega t} = j\omega \bar{B}$$

**POINT FORM:**

(1) $\nabla \times \bar{E} = -j\omega \bar{B} = -j\omega \mu \bar{H}$

(2) $\nabla \times \bar{H} = \bar{J} + j\omega \bar{D} = \sigma \bar{E} + j\omega(\varepsilon \bar{E})$

$= (\sigma + j\omega\varepsilon)\bar{E}$

(3) $\nabla \cdot \bar{D} = \rho_v$

(4) $\nabla \cdot \bar{B} = 0$

INTEGRAL FORM:

(1) $\oint \bar{E} \cdot d\bar{l} = -\int_s j\omega \bar{B} \cdot d\bar{s} = -\int_s j\omega \mu \bar{H} \cdot d\bar{s}$

$$\oint \bar{E} \cdot d\bar{l} = -j\omega\mu \int_s \bar{H} \cdot d\bar{s}$$

(2) $\oint \bar{H} \cdot d\bar{l} = \underline{I} + \int_s j\omega \bar{D} \cdot d\bar{s}$

$$= \int_s \bar{J} \cdot d\bar{s} + \int_s j\omega \varepsilon E \cdot d\bar{s}$$

$$= \int_s \sigma \bar{E} \cdot d\bar{s} + \int_s j\omega \varepsilon \bar{E} \cdot d\bar{s}$$

$$= (\sigma + j\omega\varepsilon) \int_s \bar{E} \cdot d\bar{s}$$

(3) $\oint_s \bar{D} \cdot d\bar{s} = \oint_V \rho_v \, dv$

(4) $\oint_s \bar{B} \cdot d\bar{s} = 0$

## Maxwell's equations for free space:

(7)

→ Free space is a non-conducting medium in which volume charge density, $\rho_v$ is zero and conductivity $\sigma$ is also zero.

→ point form

$$\nabla \cdot \bar{D} = 0$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t}$$

$$\nabla \cdot \bar{B} = 0$$

Integral form

$$\oint \bar{E} \cdot \bar{dL} = -\int_s \frac{\partial \bar{B}}{\partial t} \cdot \bar{ds}$$

$$\oint \bar{D} \cdot \bar{ds} = 0$$

$$\oint \bar{H} \cdot \bar{dL} = \int_s \frac{\partial \bar{D}}{\partial t} \cdot \bar{ds}$$

$$\oint_s \bar{B} \cdot \bar{ds} = 0$$

## Maxwell's equations for Good conductor:

→ conductivity is very high. So for good conductors

$$\bar{J} >> \frac{\partial \bar{D}}{\partial t}$$

$$\rho_v = 0$$

point form

$$\nabla \cdot \bar{D} = 0$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \bar{H} = \bar{J}$$

$$\nabla \cdot \bar{B} = 0$$

Integral form

$$\oint_s \bar{D} \cdot \bar{ds} = 0$$

$$\oint \bar{E} \cdot \bar{dL} = -\int_s \frac{\partial \bar{B}}{\partial t} \cdot \bar{ds}$$

$$\oint \bar{H} \cdot \bar{dL} = I = \int_s \bar{J} \cdot \bar{ds}$$

$$\oint_s \bar{B} \cdot \bar{ds}$$

Maxwell's equations for Harmonically Varying fields:

→ fields are varying harmonically with time

$$\bar{D} = \bar{D}_0 e^{j\omega t}$$ $\bar{D}_0$ max. value

$$\bar{B} = \bar{B}_0 e^{j\omega t}$$

$$\frac{\partial \bar{D}}{\partial t} = j\omega \bar{D}_0 e^{j\omega t} = j\omega \bar{D}$$

$$\frac{\partial \bar{B}}{\partial t} = j\omega \bar{B}_0 e^{j\omega t} = j\omega \bar{B}$$

Point form:

1) $\nabla \cdot \bar{D} = \rho_v$

2) $\nabla \times \bar{E} = -j\omega \bar{B} = -j\omega\mu\bar{H}$

3) $\nabla \times \bar{H} = \bar{J} + j\omega \bar{D} = \sigma \bar{E} + j\omega(\varepsilon \bar{E}) = (\sigma + j\omega\varepsilon)\bar{E}$

4) $\nabla \cdot \bar{B} = 0$

Integral form:

(1) $\oint_S \bar{D} \cdot d\bar{s} = \oint_V \rho_v \, dV$

(2) $\oint \bar{E} \cdot d\bar{l} = -\int_S j\omega \bar{B} \cdot d\bar{s}$

3) $\oint \bar{H} \cdot d\bar{l} = I + \int_S j\omega \bar{D} \cdot d\bar{s}$

$$= \int_S \bar{J} \cdot d\bar{s} + \int_S j\omega\varepsilon \bar{E} \cdot d\bar{s}$$

$$= \int_S \sigma \bar{E} \cdot d\bar{s} + \int_S j\omega\varepsilon \bar{E} \, d\bar{s}$$

$$= (\sigma + j\omega\varepsilon) \int_S \bar{E} \cdot d\bar{s}$$

(4) $\oint_S \bar{B} \cdot d\bar{s} = 0$

## Maxwell's III Equation:

$$\oint \bar{H}\cdot d\bar{l} = I_{encl}$$

$$\oint \bar{H}\cdot d\bar{l} = \int_s \left[\bar{J} + \frac{\partial \bar{D}}{\partial t}\right]\cdot d\bar{s}$$

The net mmf around a closed path is equal to the surface integral of the conduction & displacement current densities over the entire surface bounded by the closed path.

$$\nabla \times \bar{H} = \bar{J}_c + \frac{\partial \bar{D}}{\partial t}$$

## Maxwell's IV Equation:

$$\int_s \bar{B}\cdot d\bar{s} = 0$$

The net magnetic flux emerging & diverging out the closed surface is equal to zero

$$\nabla\cdot \bar{B} = 0$$



| | | Significance |
|---|---|---|
| $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$ | $\oint \bar{E}\cdot d\bar{l} = -\int_s \frac{\partial \bar{B}}{\partial t}\cdot d\bar{s}$ | Faraday's law |
| $\nabla \times \bar{H} = \bar{J}_c + \frac{\partial \bar{D}}{\partial t}$ | $\oint \bar{H}\cdot d\bar{l} = I + \int_s \frac{\partial \bar{D}}{\partial t}\cdot d\bar{s}$ | Ampere's circuital law |
| $\nabla\cdot \bar{D} = \rho_v$ | $\oint \bar{D}\cdot d\bar{s} = \int_v \rho_v\, dv$ | Gauss's law |
| $\nabla\cdot \bar{B} = 0$ | $\oint \bar{B}\cdot d\bar{s} = 0$ | No isolated magnetic charges |

# Maxwell's equation for Time-varying fields:

## Maxwell's I equation:

→ according to gauss's law, the total flux out of the closed surface is equal to the net charge within the surface.

in integral form $\int_S \bar{D}.d\bar{s} = Q_{enclosed}$

→ The net electric flux emerging or diverging out of a closed surface enclosing the volume is equal to the net charge enclosed.

$$\nabla . \bar{D} = \rho_v$$

## Maxwell's II equation:

→ Consider faraday's law which relates emf induced in a ckt to the time rate of decrease of total magnetic flux linking the ckt.

$$\oint \bar{E}.d\bar{L} = -\int_S \frac{\partial \bar{B}}{\partial t}.d\bar{s}$$

→ The net emf around a closed path is equal to the surface integral of the negative time rate of change of the magnetic flux density over the surface bounded by the closed path.

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

(9)

$$\bar{J}_D = \frac{i_D}{A}$$

$$\bar{J}_D = \frac{\varepsilon A}{A} \frac{d\bar{E}}{dt}$$

$$\bar{J}_D = \frac{d}{dt}(\varepsilon \bar{E})$$

$$\bar{J}_D = \frac{\partial \bar{D}}{\partial t}$$

→ Thus in a given medium, both the types of the currents namely the $\bar{J}_C$ & $\bar{J}_D$ may flow

hence $\bar{J}_C = \sigma \bar{E}$

$\bar{J}_D = \frac{\partial \bar{D}}{\partial t}$

$$\Rightarrow \quad \bar{J} = \bar{J}_c + \bar{J}_D$$

$$= \sigma \bar{E} + \frac{\partial \bar{D}}{\partial t}$$

$$= \sigma \bar{E} + \frac{\partial}{\partial t}(\varepsilon \bar{E})$$

Let the time dependence by $e^{j\omega t}$.

$$\bar{J} = \sigma \bar{E} + j\omega \varepsilon \bar{E}$$

→ then the ratio of magnitudes of $|\bar{J}_c|$ & $|\bar{J}_D|$

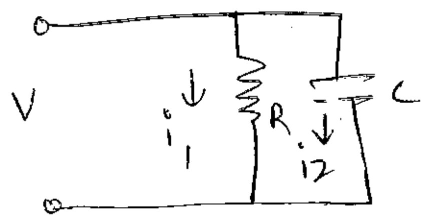
$$\frac{|\bar{J}_c|}{|\bar{J}_D|} = \frac{\sigma}{\omega \varepsilon}$$

i.e., depends on properties of the medium and the frequency

$\frac{\sigma}{\omega \varepsilon} >> 1$ medium is conductor

$\frac{\sigma}{\omega \varepsilon} << 1$ medium is dielectric

→ DISPLACEMENT CURRENT & DISPLACEMENT CURRENT DENSITY ($\bar{J}_D$)




$i_1 =$

$i_2 =$

$R =$

$C =$

$V =$

→ The current through the resistor is due to the actual motion of charges i.e., $i_1 = \frac{V}{R}$

→ this current is called conduction current, $i_c$

Conduction current density $\bar{J}_C = \frac{i_c}{A} = \sigma E$

→ Assume that the initial charge on a capacitor is zero. then for time varying voltage applied across parallel plate capacitor,

$$i_2 = C\frac{dv}{dt}$$

$$i_2 = \frac{\varepsilon A}{d}\frac{dv}{dt}$$

$$E = \frac{V}{d}$$

$$V = (d)(E)$$

$$i_D = i_2 = \frac{\varepsilon A}{d}\frac{d}{dt}(dE)$$

$$i_D = \frac{\varepsilon A}{d} d \frac{dE}{dt}$$ (as distance is not varying with time)

$$i_D = \varepsilon A \frac{dE}{dt}$$

A modification for $\nabla \times \bar{H} = \bar{J}$ is done by adding one unknown term say $\bar{N}$ (10)

$$\nabla \times \bar{H} = \bar{J} + \bar{N}$$

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} + \nabla \cdot \bar{N} = 0$$

as $\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$, to get correct conditions

$$\nabla \cdot \bar{N} = \frac{\partial \rho_v}{\partial t}$$

$$\nabla \cdot \bar{D} = \rho_v$$

$$\nabla \cdot \bar{N} = \frac{\partial}{\partial t}(\nabla \cdot \bar{D})$$

$$= \nabla \cdot \frac{\partial \bar{D}}{\partial t}$$

on comparing $\bar{N} = \frac{\partial \bar{D}}{\partial t}$

$$\therefore \quad \nabla \times \bar{H} = \bar{J}_c + \frac{\partial \bar{D}}{\partial t}$$

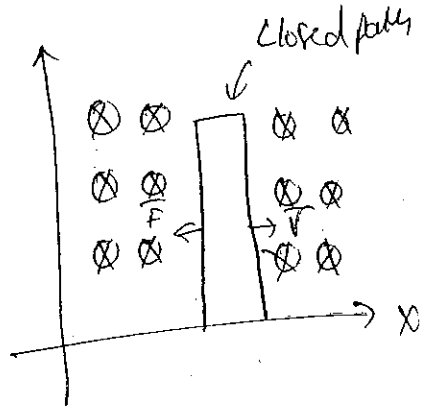


→ The I term in the above equation is conduction current density. which indicates that the current is due to the moving charges.

→ The II term in the equation represents current density expressed in $A/m^2$. And this is called as displacement current density. $\bar{J}_D$

$$\therefore \quad \nabla \times \bar{H} = \bar{J}_c + \bar{J}_D$$

2) moving closed path in static $\bar{B}$




$$\bar{F} = Q\bar{V}\times\bar{B}$$

$$\bar{E}_m = \frac{\bar{F}}{Q} = \bar{V}\times\bar{B}$$

$$\oint \bar{E}_m\cdot d\bar{L} = \oint(\bar{V}\times\bar{B})\cdot d\bar{L}$$

3) both induced emf = self emf + Motional emf

$$\oint \bar{E}\cdot d\bar{L} = \int \frac{\partial \bar{B}}{\partial t}\cdot d\bar{S} + \oint (\bar{V}\times\bar{B})\, d\bar{L}$$



**Modified ampere's circuital law for time varying fields:**

$$\nabla\times\bar{H} = \bar{J}$$

$$\nabla\cdot(\nabla\times\bar{H}) = \nabla\cdot\bar{J}$$

but $\nabla\cdot(\nabla\times\bar{H}) = 0$

$$\nabla\cdot(\nabla\times\bar{H}) = \nabla\cdot\bar{J} = 0 \rightarrow (1)$$

current continuity equation

$$\nabla\cdot\bar{J} = -\frac{\partial\rho_v}{\partial t} \rightarrow (2)$$

equation (2) becomes true when $\frac{\partial\rho_v}{\partial t} = 0$

$\therefore$ (1) & (2) equations are not compatible for time varying fields.

→ Applying divergence theorem is then applied to the LHS thus converting the volume integral into surface integral the encloses the volume.

$$-\oint_{area} (\bar{E} \times \bar{H}) \cdot d\bar{s} = \int_{vol} \bar{J}_c \cdot \bar{E}\, dv + \frac{d}{dt} \int_{vol} \frac{1}{2} \bar{D} \cdot \bar{E}\, dv + \frac{d}{dt} \int_{vol} \frac{1}{2} \bar{B} \cdot \bar{H}\, dv$$

The above equation is known as poynting's theorem.

⇒ on the RHS ⇒ I integral is the ohmic power dissipated within the volume

⇒ II integral is the total energy stored in the electric field

⇒ III integral is the total energy stored in the magnetic field.

⇒ Since the time derivatives are taken of the II & III integrals those results give the time rates of increase of energy stored within the volume, or the instantaneous power going to increase the stored energy.

⇒ The sum of the expressions on the right must therefore be the total power flowing into this volume, and so the total power flowing out of the volume is

$$\oint_{area} (\bar{E} \times \bar{H}) \cdot d\bar{s} \quad \text{watts}$$

where the integral is over the closed surface surrounding the volume. The cross product $\bar{E} \times \bar{H}$ is known as the poynting vector, S. $\boxed{S = \bar{E} \times \bar{H} \quad W/m^2}$ ← instantaneous power density

POYNTING THEOREM:

→ In order to find the power flow associated with an electromagnetic wave, it is necessary to develop a power theorem for the electromagnetic field known as the poynting theorem.

→ from Maxwell's III equation for time varying fields

$$\nabla \times \bar{H} = \bar{J}_c + \frac{\partial \bar{D}}{\partial t}$$

Scalar product on both sides with $\bar{E}$

$$\bar{E} \cdot \nabla \times \bar{H} = \bar{E} \cdot \bar{J}_c + \bar{E} \cdot \frac{\partial \bar{D}}{\partial t}$$

according to vector identity

$$\nabla \cdot (\bar{E} \times \bar{H}) = -\bar{E} \cdot \nabla \times \bar{H} + \bar{H} \cdot \nabla \times \bar{E}$$

substituting in above equation

$$\bar{H} \cdot \nabla \times \bar{E} - \nabla \cdot (\bar{E} \times \bar{H}) = \bar{J}_c \cdot \bar{E} + \bar{E} \cdot \frac{\partial \bar{D}}{\partial t}$$

we know $\quad \nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

$$\therefore \quad -\bar{H} \cdot \frac{\partial \bar{B}}{\partial t} - \nabla \cdot (\bar{E} \times \bar{H}) = \bar{J}_c \cdot \bar{E} + \bar{E} \cdot \frac{\partial \bar{D}}{\partial t}$$

$$-\nabla \cdot (\bar{E} \times \bar{H}) = \bar{J}_c \cdot \bar{E} + \varepsilon \bar{E} \cdot \frac{\partial \bar{E}}{\partial t} + \mu \bar{H} \cdot \frac{\partial \bar{H}}{\partial t}$$

$$\Downarrow \qquad \qquad \Downarrow$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2} \bar{D} \cdot \bar{E}\right) \qquad \frac{\partial}{\partial t}\left(\frac{1}{2} \bar{B} \cdot \bar{H}\right)$$

$$\boxed{\begin{array}{l} \frac{\partial}{\partial t}(\bar{E} \cdot \bar{E}) = \frac{\partial \bar{E}}{\partial t} \cdot \bar{E} + \bar{E} \cdot \frac{\partial \bar{E}}{\partial t} = 2\bar{E} \cdot \frac{\partial \bar{E}}{\partial t} \\ \frac{\partial}{\partial t}(\bar{E}^2) = 2\bar{E} \cdot \frac{\partial \bar{E}}{\partial t} \Rightarrow \frac{1}{2} \frac{\partial}{\partial t}(\bar{E})^2 = \bar{E} \cdot \frac{\partial \bar{E}}{\partial t} \end{array}}$$

$$\therefore \quad -\nabla \cdot (\bar{E} \times \bar{H}) = \bar{J}_c \cdot \bar{E} + \frac{\partial}{\partial t}\left(\frac{1}{2} \bar{D} \cdot \bar{E}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2} \bar{B} \cdot \bar{H}\right)$$

integrating throughout a volume

$$-\int_{vol} \nabla \cdot (\bar{E} \times \bar{H})\, dv = \int_{vol} \bar{J}_c \cdot \bar{E}\, dv + \int_{vol} \frac{\partial}{\partial t}\left(\frac{1}{2} \bar{D} \cdot \bar{E}\right) dv + \int_{vol} \frac{\partial}{\partial t}\left(\frac{1}{2} \bar{B} \cdot \bar{H}\right) d$$

UNIT - V - problems

1) In a material for which $\sigma = 5.0$ S/m and $\varepsilon_r = 1$, the electric field intensity is $E = 250 \sin 10^{10} t$ V/m. Find the conduction & displacement current densities and the freq at which both have equal magnitude.

Ans)

$$J_C = \sigma E$$

$$= 5 (250 \sin 10^{10} t)$$

$$= 1250 \sin 10^{10} t \quad A/m^2$$

$$J_D = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t} (\varepsilon E)$$

$$= \frac{\partial}{\partial t} (\varepsilon_0 \varepsilon_r \bar{E})$$

$$= \frac{\partial}{\partial t} (8.854 \times 10^{-12} \times 1 \times 250 \sin 10^{10} t)$$

$$= 22.135 \cos 10^{10} t \quad A/m^2$$

$$\frac{\bar{J}_c}{\bar{J}_D} = \frac{\sigma}{\varepsilon \omega} = 1$$

$$\omega = \frac{\sigma}{\varepsilon} = \frac{5}{8.854 \times 10^{-12} \times 1} = 5.6471 \times 10^{11}$$

$$\omega = 2\pi f.$$

$$f = \frac{\omega}{2\pi} = \frac{5.6471 \times 10^{11}}{2\pi} = 89.87 \text{ GHz}$$